A TREATISE

ON

NAUTICAL ASTRONOMY

JOHN MERRIFIELD LL.D.

A TREATISE

ON

NAUTICAL ASTRONOMY

FOR THE USE OF STUDENTS.

BY

JOHN MERRIFIELD, LL.D., F.R.A.S., F.R. MET. SOC.,

HEAD MASTER OF THE NAVIGATION SCHOOL, PLYMOUTH.

AUTHOR OF "NAVIGATION AND NAUTICAL ASTRONOMY;" "MAGNETISM, AND THE DEVIATION OF THE COMPASS;" "HAND-BOOK TO THE BOARD OF TRADE EXAMINATIONS;" "A TREATISE ON NAVIGATION, FOR THE USE OF STUDENTS," ETC.

London:

SAMPSON LOW, MARSTON, SEARLE, AND RIVINGTON,

CROWN BUILDINGS, 188, FLEET STREET.

1886.

PREFACE.

THE favourable notices my former works received from the Press and from my fellow-teachers have induced me to offer another contribution to the literature of that subject in the teaching of which I have been engaged for nearly twenty-five years.

The present work is offered as a companion volume to the "Treatise on Navigation," in order that the subject of Nautical Astronomy may be presented to the student in a manner demanded by the different Examining Boards without descending to a cram-book: and I hope the experience I have acquired will be of service to those who have to undergo the ordeal of examination. Not only is the explanation of every proposition given for the general reader, but the mathematical proof is also added for the student; and many problems are discussed which belong rather to Physical Astronomy than to the subject under consideration. This is rendered necessary because questions bearing on these problems have from time to time been introduced into examination papers; but any one tolerably well read in Spherical Trigonometry will find no difficulty in mastering the volume. At the end of each chapter a very full collection of examples is given relating to the preceding matter. Many have been gleaned from those set by the Admiralty, at the Universities, and by the Science and Art Department, the latter being marked E., A., or Honours, with the year attached in which they were proposed; and the sources from which others (not original) are derived, are given at the end of the question. It is hoped that the nature of the Examination by each Board will thus be sufficiently indicated; so that the student may prepare himself accordingly. As the aim of such a work is to give methods for actual practice in finding the position of the

observer on the earth's surface and for discovering the errors of the instruments he uses, a very large number of questions requiring numerical solutions has been added. These have been computed for the year 1887. As the "Nautical Almanac" is absolutely necessary for sea purposes, as well as for the solution of the questions here proposed, I think it will be beneficial for the student to early acquire rapidity and accuracy (so essential in a good computer) in reducing data taken from it; and this can only be attained by the actual use of the work itself. I have, therefore, omitted all data which can be derived from that source.

In the preparation of this volume I have consulted every work on the subject with which I am acquainted, and must express my obligation to "Woodhouse's Astronomy," "Godfrey's Astronomy," "Loomis' Astronomy," "Riddle's Navigation," and to many others for the very valuable suggestions conveyed by them. In "Clearing the Lunar Distance" I have introduced a method suggested by Sir G. B. Airy, F.R.S., the late Astronomer-Royal, and I here tender him my sincere thanks for his ready permission to use it in this work. I also present the reader with a method of my own for the same purpose, which has already appeared in the "Monthly Notices of the Royal Astronomical Society" for April, 1884. This one is independent of case, requires no special tables, and is direct in its application; and, being a very close approximation, is therefore well adapted for use at sea, especially when we consider that on a voyage all the data employed are only approximations.

To my assistants, Mr. Charles Morris, F.R. Met. Soc., and my son, Mr. W. Venner Merrifield, B.A. Cantab., I owe a debt of gratitude for bestowing so much time in checking the calculations and for other valuable assistance; and here I gladly acknowledge how much I owe to them. Every care has been taken to render the work full and accurate; but where so many figures are employed, slips may probably occur. I shall, therefore, be thankful to my fellow-teachers and students for any corrections in the computations, as well as for hints for the improvement of the work.

J. M.

NAVIGATION SCHOOL, PLYMOUTH,
April, 1886.

CONTENTS.

CHAPTER XI.

CHAPTER XII.

CHAPTER XIII.

CHAPTER XIV.

CHAPTER XV.

CHAPTER XVI.

CHAPTER XVII.

CHAPTER XVIII.

NAUTICAL ASTRONOMY.

CHAPTER I

Second method of finding a ship's position—Instruments used—Definitions.

THE second and most accurate method for finding the place of a ship at sea is by determining the situation of heavenly bodies with respect to the horizon or to each other by means of instruments; then from their known positions with regard to certain co-ordinates, and by rules deduced from spherical trigonometry, the ship's place is calculated. The instruments used by the mariner for this purpose are the *Sextant* or *Quadrant*, the *Artificial Horizon*, and the *Chronometer*. This method constitutes *Nautical Astronomy*, because a knowledge of astronomy is requisite for the calculations employed; and it may be defined as the application of astronomy to navigation. When the position of the ship has been calculated by their means, we say it has been determined "by observation," to distinguish it from position "by dead reckoning," as explained in the "Treatise on Navigation."

Before proceeding the following definitions must be thoroughly understood:—

A SPHERE OR GLOBE is a solid body, every part of whose surface is equally distant from a point fixed relatively to that surface. This point is the centre of the sphere. It may be conceived to be generated by the revolution of a semicircle about its diameter, which remains fixed in space.

A GREAT CIRCLE is one whose plane passes through the centre of the sphere. If any one great circle be selected as the primary, then all others at right angles to this one are called secondaries, and a point in either of these 90° from the primary is its pole.

A great circle on the sphere is analogous to the straight on the plane.

A SMALL CIRCLE is one whose plane does not pass through the centre of the sphere.

THE CELESTIAL CONCAVE.—The hollow spherical surface on which all the heavenly bodies appear to be projected is immeasurably distant, and is called the *celestial concave*, or the *heavens*.

THE AXIS OF THE HEAVENS.—From the diurnal rotation of the earth on its axis the heavens appear to turn round from east to west in a day, on a fixed line passing through the spectator and parallel to the axis of the earth. This line is the axis of the heavens, and from the immense distance of the celestial concave is generally considered to coincide with the axis of the earth produced.

The POLES OF THE HEAVENS are those points in the celestial concave where the axis of the heavens is conceived to meet its surface. Now, although the earth moves in an orbit whose diameter is about 185,400,000 miles, its axis is always parallel to any position it formerly had, and when produced it describes an ellipse in the heavens equal to the earth's orbit. Still this immense distance vanishes when compared with the radius of the celestial concave; hence we say the poles of the heavens are determined by producing the axis of the earth to meet the celestial concave. They are called north and south similar to those on the terrestrial sphere.

CELESTIAL EQUATOR OR EQUINOCTIAL is that great circle in the heavens equally distant from the two celestial poles. It may be conceived to be that great circle formed by producing the plane of the terrestrial equator to meet the celestial concave. Its plane will therefore be at right angles to the axis of the heavens. We shall use the term equinoctial to prevent ambiguity, and shall restrict equator to the terrestrial one.

The ECLIPTIC is that great circle in the heavens traced out by the sun's apparent path from west to east among the fixed stars. As the earth makes a complete revolution in her orbit once in a year, the sun appears projected back on the celestial concave among the fixed stars. The line traced out by the apparent place of the sun on the celestial concave as seen by a spectator at the earth's centre is the great circle called the *ecliptic*.

OBLIQUITY OF THE ECLIPTIC is the angle included between the equinoctial and the ecliptic; it amounts to about 23° 27′ 6″.

TRUE ZENITH is the point in the celestial concave where a line perpendicular to standing water (the normal) at the place of observation meets the heavens. If this line be produced backwards to meet the plane of the equator, the angle thus made is the *true latitude*, or *latitude by observation* of the place of the observer.

REDUCED ZENITH is the point in the celestial concave where a radius of the earth to the observer produced meets the heavens. The angle included between this radius and the plane of the equator is the *reduced latitude* of the place of observation. The *angle of the vertical* is the difference between the true and reduced latitudes. (See page 149 *et seq.*, "Treatise on Navigation.")

ZENITH DISTANCE of a celestial object is the angle at the plane of the equinoctial between the observer's true zenith and the object.

The VISIBLE HORIZON is the boundary of vision in the open sea; and because the earth is spheroidal, the radius of the visible horizon must become greater as the observer ascends above the surface.

The SENSIBLE HORIZON is the tangent plane at the observer's feet produced to meet the celestial concave.

The RATIONAL HORIZON is the plane through the earth's centre parallel to the sensible horizon produced to meet the heavens.

THE CELESTIAL HORIZON.—If the planes of the sensible and rational horizons be produced, they will, from the great distance of the celestial concave, meet the heavens in a great circle. This is the *celestial horizon.*

From the above definitions it will be seen that the zenith is the pole of the celestial horizon. The point in the heavens diametrically opposite to the zenith is the *nadir.*

VERTICAL CIRCLES, AZIMUTH CIRCLES, or CIRCLES OF ALTITUDE are great circles passing through the zenith, and hence they cut the horizon at right angles.

CELESTIAL MERIDIAN is that vertical circle which passes through the observer's zenith and the poles of the heavens. It may be conceived to be described by producing the plane of the terrestrial meridian to meet the celestial concave.

NORTH AND SOUTH POINTS OF THE HORIZON are the two points where the meridian and the horizon meet.

HOUR CIRCLE, or CIRCLE OF DECLINATION of a celestial object, is that great circle which passes through the elevated pole and through the object.

PRIME VERTICAL is that vertical circle whose plane is at right angles to that of the meridian. Where it cuts the horizon are its east and west points.

DECLINATION of a celestial body is the arc of an hour circle intercepted between the object and the equinoctial; or it is the perpendicular distance of an object from the equinoctial measured on an hour circle. It is the angle at the centre of the earth subtended by the arc of an hour circle intercepted between the equinoctial and the object.

POLAR DISTANCE is the angle at the centre of the earth subtended by the arc of an hour circle between the object and the elevated pole. It will hence be seen that the polar distance of an object is the complement of its declination.

CELESTIAL LATITUDE is the distance of an object from the ecliptic measured on a great circle perpendicular to it.

CIRCLES OF LATITUDE are great circles in the heavens perpendicular to the ecliptic, hence all meet in the poles of the ecliptic.

EQUINOXES.—The sun in his apparent annual path crosses the equinoctial twice; and because all great circles bisect one another, these occur at intervals of six months; first about the 21st of March, and secondly about the 23rd of September. The *times of crossing* are called the *equinoxes*, the former the vernal, the latter the autumnal. The term is derived from the fact that at these times the nights are of equal length all over the world.

FIRST POINT OF ARIES is the point of intersection of the equinoctial and the ecliptic when the sun changes his declination from south to north. Hence it is the point in the equinoctial corresponding to the sun's place at the time of the vernal equinox.

PRECESSION OF THE EQUINOXES.—Owing to the spheroidal form of the earth, the effects of the attraction of the sun and moon on the part protruding beyond the sphere is to cause the plane of the equator to retrograde along the ecliptic, whilst the latter remains stable. This retrogression of the first point of Aries, amounting to 50·2″ annually, is called *the precession of the equinoxes.*

The SOLSTICES are the two periods of the year when the sun attains his greatest declination. These occur about June 22nd and December 22nd, and are called respectively the summer and winter solstice. The points in the ecliptic where the sun is at these times are called the solstitial points.

COLURES are great circles which pass through either the equinoctial or solstitial points and the poles of the heavens.

RIGHT ASCENSION is the arc of the equinoctial measured eastward from the first point of Aries to where the hour circle through the object meets it; or it is the angle at the pole between the hour circle through the object and that through the first point of Aries. It is recorded in time, hours, minutes, and seconds, from 0 to 24 hours, or in arc degrees, &c., from 0 to 360°.

CELESTIAL LONGITUDE is the arc of the ecliptic measured eastward from the first point of aries to where a great circle perpendicular to the ecliptic and passing through the object meets it. Celestial longitude, like right ascension, is measured through the whole circle, that is through 360° or 24 hours.

The difference between declination and the celestial latitude of an object, and between its right ascension and its celestial longitude, is that declination and right ascension are referred to the equinoctial and to the vernal equinoctial colure as co-ordinates; whilst celestial latitude and longitude are referred to the ecliptic, and to a great circle perpendicular to it through the first point of aries as co-ordinates. Knowing the position of a body expressed in one set of co-ordinates, its position expressed in the other set can easily be deduced.

OBSERVED ALTITUDE is the angle at the eye subtended by the height of an object above the visible horizon, measured on a vertical circle.

APPARENT ALTITUDE is the angle at the earth's surface subtended by the height of the object above the sensible horizon measured on a vertical circle.

TRUE ALTITUDE is the angle at the centre of the earth subtended by the height of the object above the rational horizon measured on a vertical circle. From these definitions it will be seen that the true altitude is the complement of the true zenith distance.

AZIMUTH of a heavenly body is the angle at the zenith between the vertical circle through the object and the meridian of the observer; or, it is the angle at the place subtended by an arc of the horizon between the points where the meridian of the observer and the vertical circle through the object meets the horizon.

AMPLITUDE of a heavenly body is the angle at the zenith between the vertical circle through the object at rising or setting

and the prime vertical; or, it is the angle at the place of observation subtended by an arc of the horizon between the object when rising or setting and the east or west point. The term amplitude is applied only to the bearing of objects when rising or setting.

HOUR ANGLE of an object is the angle at the elevated pole between the meridian of the observer and the hour circle through the object.

TRANSIT is the term used to denote the passage of an object across the meridian; or the passing of one object over another.

PERIGEE is the point in a body's orbit nearest the earth.

APOGEE is the point in a body's orbit farthest from the earth.

PERIHELION is the point in a body's orbit nearest the sun.

APHELION is the point in a body's orbit farthest from the sun.

The HELIOCENTRIC POSITION of an object is its position on a celestial sphere which has the sun for its centre.

The GEOCENTRIC POSITION of an object is its position on a celestial sphere which has the earth for its centre.

CHAPTER II.

Co-ordinates — Co-ordinates on the celestial sphere — Projection— Gnomonic, orthographic, and stereographic projections—Right sphere—Parallel sphere—Oblique sphere—Illustrations—Exercise —Examination.

CO-ORDINATES AND PROJECTION.

"CO-ORDINATES are a set of lines, angles, or planes, or a combination of these, which, taken together, define the position of the several points of a given surface, or points in space."[1] This method of defining position was invented by Descartes. They may be either *rectangular*, *polar*, or *oblique.* Rectangular when the point is defined by reference to lines at right angles to one another; polar when the point is defined by the length of a line and the angle swept out in a given direction by that line from a given initial position; and oblique when the point is defined by lines making known (not right) angles with each other.

CO-ORDINATES ON THE CELESTIAL SPHERE.—These are divided into three systems, viz. the *equinoctial system*, the *horizon system*, and the *ecliptic system* :—

(*a*) In the *equinoctial system*, with rectangular co-ordinates, the lines of reference are the vernal equinoctial colure and the equinoctial line: the co-ordinates are right ascension and declination. With polar co-ordinates the initial position is the meridian, the length of the line is polar distance, and the angle swept out is the hour angle.

(*b*) In the *horizon system*, with rectangular co-ordinates, the lines of reference are the horizon and meridian: the co-ordinates are azimuth and altitude.

(*c*) In the *ecliptic system*, with rectangular co-ordinates, the lines of reference are the ecliptic and a great circle through the

[1] Harboard's Glossary.

first point of aries and the pole of the ecliptic: the co-ordinates are longitude and latitude.

PROJECTION.

In every branch of mathematics a well-drawn figure is a great help to the student's thorough understanding of the problem he has to solve: and in our present subject great service is rendered by a knowledge of projection.

PROJECTION is the art of delineating solid bodies on any surface according to certain definite laws. In Nautical Astronomy the sphere is the only body whose projection is required, and a plane the only surface on which the projection is made. The plane used is that of the great circle which is at right angles to a line joining the eye of the supposed observer with the centre of the sphere. This great circle is called the *primitive circle.*

The three chief projections of the sphere are the *gnomonic*, the *orthographic*, and the *stereographic.*

The GNOMONIC PROJECTION has already been explained, and its use illustrated in the chapter on Great Circle Sailing in the "Treatise on Navigation."

The ORTHOGRAPHIC PROJECTION shows the appearance the lines on the sphere would have if viewed from an infinite distance; and, therefore, the rays of light would all be parallel. It may be illustrated thus:—Let $A\ D\ B$ be the section of a hemisphere at right angles to its plane surface, which is the primitive.

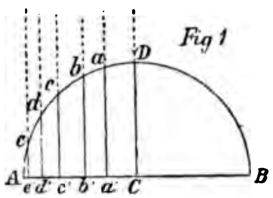


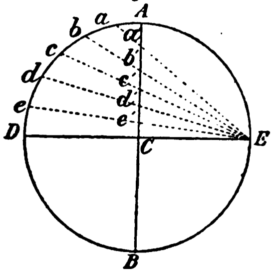
Fig 1

Divide $D\ A$ into any number of equal parts, say six, then each one will subtend an angle at the centre of 15°, and to an eye at an infinite distance a will appear projected at a', b at b', c at c', and so on; and thus $C\ a'$, $a'\ b'$, $b'\ c'$, &c., are projections of arcs subtending 15° at the centre on the primitive. From the figure it is at once deduced that if r be the radius of the sphere $C\ a' = r'$ sin. 15°, $a'\ b' = r$ (sin. 30° − sin. 15°); $b'\ c' = r$ (sin. 45° − sin. 30°), and so on. Thus it may be proved, or it can be seen from the figure that $C\ a'$ is much nearer the true length of an arc subtending 15° than $e'\ A$ is, being in the proportion of r sin. 15° to r (sin. 90° − sin. 75°): and hence it is shown that on this projection the lines near the

centre of the primitive are a much closer approximation to the true length than those near the circumference: and all lines in the latter position are therefore much distorted.

The STEREOGRAPHIC PROJECTION gives us the appearance the lines on the sphere would have if viewed from a point in the circumference of the sphere; and looking at the opposite concave hemisphere, the primitive plane between being supposed transparent. Let the eye be placed at E, then points and lines on the concave hemisphere $A\ D\ B$ will appear projected on the plane of projection (primitive) $A\ C\ B$. Divide $D\ A$ as before into six equal parts, each representing 15°, as a, b, c, d, e, and join $E\ a$, $E\ b$, $E\ c$, &c., meeting the primitive in a', b', c', d', e'; these latter points are the projections of a, b, c, d, e, on the stereographic projection. From the figure $C\ e'$ is the tangent of half the angle $e\ C\ D$ multiplied by the radius or r tan. $7\frac{1}{2}°$; $e'\ d'$ is r (tan. $15°$ − tan. $7\frac{1}{2}°$), and so on: the lines near the centre being much less than corresponding ones near the circumference of the primitive, which approximate to the actual length. In this projection, therefore, lines and figures near the centre are distorted. The lengths $C\ e'$, $C\ d'$, $C\ c'$, &c., are called the subtangents of 15°, 30°, 45°, &c.

Fig 2.



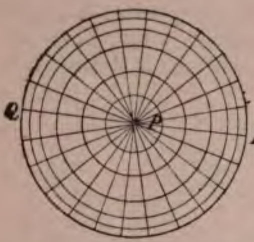
According to the position the poles of the heavens take in the projection, the celestial sphere has received different names: thus it is called:—

(*a*) A RIGHT SPHERE when the primitive is the meridian; and, therefore, contains the zenith and poles. The extremities of the vertical diameter are the zenith and nadir, the horizontal diameter represents the horizon, the vertical diameter is the prime vertical, the centre is the east or west point of the horizon. All other great circles drawn from the zenith to the horizon will appear as semi-ellipses, and represent vertical circles, azimuth circles, or circles of altitude. The elevated pole must be taken on the primitive at an angle from the horizon equal to the latitude (to be proved hereafter); a diameter through this pole is the axis; great circles from pole to pole, shown as semi-ellipses, are hour circles, and a diameter at right angles to

the axis is the equinoctial. The angles at the poles between the meridian and the hour circles are hour angles, and the angles at the zenith between the meridian and any vertical circle is the azimuth of a body on that circle. (See Fig. 4.)

(*b*) A PARALLEL SPHERE when the primitive is the equinoctial, and is so called because the celestial bodies in their revolutions move parallel to the equinoctial or primitive circle. The centre *P* is the pole, the radii are meridians drawn at 15° interval, and the concentric circles within are parallels of declination drawn at every 15° from the equinoctial; and as the equinoctial also represents the horizon the inner circles also represent parallels of altitude. The sphere as thus shown is particularly adapted to illustrate the polar regions of either the celestial or terrestrial spheres; but it is seldom used in Nautical Astronomy.

Fig 3.



(*c*) An OBLIQUE SPHERE when the primitive is the horizon, the centre the zenith, the vertical diameter the meridian, the horizontal one the prime vertical, the radii are vertical circles, the angle at the centre contained between either of them and the vertical diameter is the azimuth of a body situate on the particular vertical circle. In this delineation of the sphere, the pole is in the vertical diameter between the centre and the circumference: the great circles through the pole are hour circles, and except the meridian, all meet the primitive obliquely. (See Fig. 5.)

The stereographic projections of the right and of the oblique spheres are the ones chiefly used to illustrate the problems in Nautical Astronomy; the former is (as already shown) on the plane of the meridian, and the latter on the plane of the horizon. To gain a clear conception of the former, the student should place a globe (on which the lines of the sphere are drawn) in such a position that the wooden horizon may be level with the eye, the brass meridian as nearly north and south as possible, and therefore in the plane of the meridian, with the pole elevated to the angle equal to his latitude. Then if he moves back a short distance east or west of the globe he will gain an accurate notion of the relative positions of the lines on the sphere

on this projection. For the oblique sphere he must allow the globe to remain in the same position, but must now carry his eye vertically over the centre of the globe; and in this position he ought to experience no difficulty in tracing the lines as they appear. In each of these projections the lines on the surface of the sphere are distorted, those which are near to the centre of the primitive being less so than those farther distant; but still they are shown as they appear to the eye. After the names and the positions of these lines are well known, the student should trace them on the celestial concave until perfect.

The following diagrams illustrate the definitions on the stereographic projections on the two planes.

Fig 4.

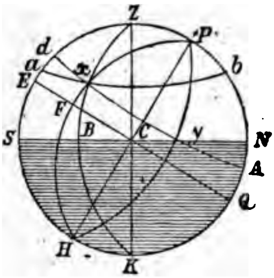
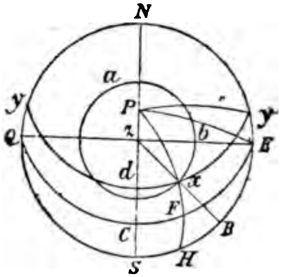


Fig 5.



We have represented the sun when his altitude is 38°, the latitude of the observer 60° N., and declination 15° N., and therefore the hour angle about 2 hrs. 23 m., and azimuth S. 46° E., it being before noon.

Fig. 4.	Fig. 5.	
NKSZ	*NESQ*	are the primitive circles.
NCS	*NESQ*	represents the horizon.
Z	*Z*	,, ,, zenith.
K		,, ,, nadir.
ECQ	*ECQ*	,, ,, equinoctial.
N	*N*	,, ,, north point of the horizon.
S	*S*	,, ,, south ,, ,, ,,
C	{ *E*	,, ,, east ,, ,, ,,
	{ *Q*	,, ,, west ,, ,, ,,
ZC	*QZE*	,, ,, prime vertical.
PCH	*QPE*	,, ,, six o'clock hour circle.
x	*x*	,, ,, position of the body as the sun.
NPZS	*NPZS*	,, ,, meridian of the observer.

Fig. 4.	Fig. 5.	
xB	xB	represents the sun's altitude.
xZ	xZ	" " " zenith distance.
xF	xF	" " " declination.
xP	xP	" " " polar distance.
axb	axb	" " parallel of altitude through x.
xd	xd	" " sun's meridianal distance.
d	d	" " " position at noon.
Sd	Sd	" " " altitude "
Zd	Zd	" " " meridian zenith distance.
PxH	PxH	" " hour circle through the sun.
xPS	xPS	" " sun's hour angle or apparent time from noon.
ZxB	ZxB	" " azimuth circle through the sun.
SB SZB	$\left.\begin{matrix}SB\\SZB\end{matrix}\right\}$	" " sun's azimuth from south.
NB NZB	$\left.\begin{matrix}NB\\NZB\end{matrix}\right\}$	" " " " " north.
y	y	" " " place at rising.
Cy	Ey	" " " amplitude from the east.
A		" " " place at midnight.
Ayd	ydy'	" " " path in the heavens.
dPy	dPy	" " " time from noon of the sun's rising, or half the length of the day.
APy		" " " half the length of the night.

EXERCISE I.

1. Distinguish between Navigation and Nautical Astronomy? What are the chief instruments used in each?

2, Define *horizon*, *azimuth*, *apparent altitude*, and *latitude* of a heavenly body. Show how the position of a heavenly body may be determined by referring it, with the aid of another great circle, (*a*) to the equinoctial; (*b*) to the horizon? *E.* 1869.

3. Give definitions of *ecliptic*, *celestial meridian*, *declination*, and *amplitude* of a heavenly body? Explain your definitions by figures. *E.* 1871.

4. Explain, with diagrams, the terms *right ascension*, *prime vertical*, *obliquity of the ecliptic*. Define *equinox* and *solstice*. *Royal Naval College*, 1873.

5. Define *circles of declination*, *azimuth circles*, *sensible and rational horizon*, also *precession of the equinoxes*, and illustrate by diagrams. *A.* 1872.

6. Explain the terms in Nautical Astronomy, *heliocentric* and *geocentric* position of a heavenly body. *Royal Naval College*, 1872.

7. Give definitions of the following terms: *latitude, reduced latitude, longitude, hour angle,* and *circle of declination.*

Royal Naval College, 1864.

8. What is a definition? What are the *solstices,* what the *ecliptic,* and what the *first point of aries?*

Royal Naval College, 1865.

9. Define *parallels of declination, altitude of a heavenly body, hour circle,* and *prime vertical.* Illustrate with diagrams. Show how the position of a point may be fixed by its polar distance and its hour angle. *E.* 1875.

10. Give definitions of the following terms (without using diagrams): *right ascension, declination, latitude, longitude, azimuth, amplitude, polar distance,* and *hour angle* of a heavenly body. Afterwards explain each term by means of a diagram.

Royal Naval College, 1864.

11. What do you mean by *celestial concave?* Explain the term "angular distance of two heavenly bodies." Define and illustrate by diagrams the following terms: *celestial meridian, vertical circles, reduced zenith,* and *equinoctial points. E.* 1876.

12. How is the position of a star determined?

B.A. and B.Sc. London, 1865.

13. What do you mean by co-ordinates? How many systems of co-ordinates are used in fixing the position of a point in the celestial sphere? Explain them.

14. What do you mean by projection? Illustrate and explain the chief projections used in Nautical Astronomy. What do you mean by a *right sphere,* a *parallel sphere,* and an *oblique sphere?* On what plane is each drawn?

15. How is the place of a heavenly body defined with reference (a) to the zenith and horizon of an observer, (b) to the equinoctial and pole of the heavens? Distinguish between the latitude and longitude of a place on the earth's surface, and the latitude and longitude of a heavenly body. *E.* 1881.

16. Define the equator (equinoctial) and ecliptic; and supposing that each consisted of a ring of stars, describe the apparent diurnal motions to a spectator in London.

B.A. and B.Sc. London, 1861.

17. Distinguish between the position of a ship "by observation" and her position "by dead reckoning." Define and illustrate colures, true altitude, transit, apogee, and perihelion.

18. Show by a diagram how the latitude and longitude

enable you to determine the position of a place. Why is the meridian altitude the greatest during the day? *E.* 1877.

19. What is the right ascension and declination of a heavenly body? Draw a figure indicating the position of a star of which the N. P. D. is 30°, and the R. A. 3 hours.

B.A. and B.Sc. London, 1870.

20. Given the R. A. and declination of a heavenly body, and the obliquity of the ecliptic, show how to obtain the formulæ for computing its latitude and longitude.

Example: Given the latitude and longitude of the moon 2° 48′ 45″ N. and 63° 54′ respectively, find her declination and right ascension. *For Beaufort Testimonial,* 1867.

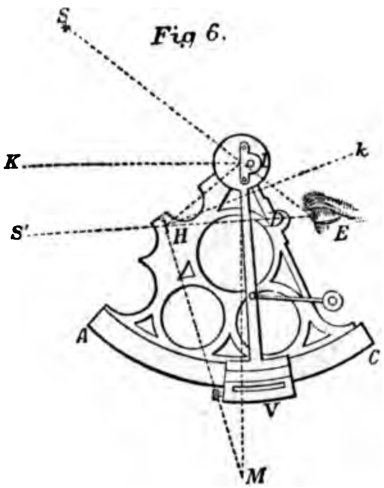
CHAPTER III.

Instruments—Sextant, proof of principle involved—The vernier, proof for its subdivision—Adjustments of the sextant—Proof for error caused by collimation—Index error, how caused, how found, proof for its application—Artificial horizon, proof for its application—Artificial horizon by the author—The chronometer, its error and rate, how found—Exercise—Examination.

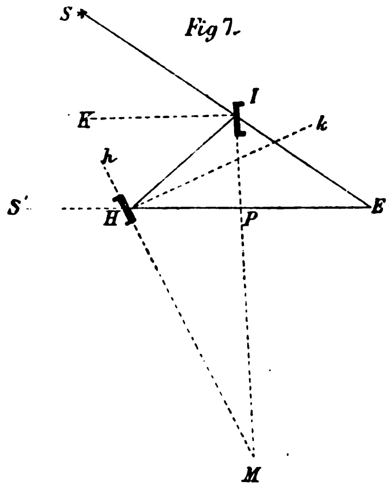
THE chief instruments used in Nautical Astronomy are the quadrant or sextant, the artificial horizon, and the chronometer.

THE SEXTANT AND QUADRANT.—In classing these together it is not to be supposed they are one and the same instrument; but they are both constructed on the same principle, and an explanation of one will almost in every respect serve for the other. They are instruments for taking angular measurements; that is for measuring angles at the eye subtended by the line joining any two points, provided that for the sextant they are not more than about 120°, and for the quadrant about 90° apart. To make these measurements the instrument must be held so that its plane passes through the two objects whose angular distance is required, and the vernier moved along over the arc until the reflection of one object coincides with the other. The angle is then, by aid of the vernier, read off from the arc or limb.

The description and name and use of each part may be best learned from the instrument itself; and the student, if he has not one of his own, is strongly recommended to procure the loan of one for the purpose. *I* is a mirror called the index-glass, *H* the horizon-glass, *A C* the arc or limb, *V* the vernier at the end of the index-bar *I V*, and a telescope is screwed into the collar at *D*, which can be moved farther from or towards the plane of the instrument by the large milled-head screw called the up-and-down screw, acting through the up-and-down piece. The use of this motion is to cause two objects when brought together to appear of the same brightness. In taking observations for altitude the sextant is held vertically, and the eye placed at the telescope *E*; the horizon-glass *H* is directed to a point *S'*



in the horizon on a vertical circle through S. Conceive E joined with S and S', then $S E S'$ is the angular distance on a vertical circle through S, between it and the horizon; that is, it is the observed altitude of the object S. A beam of light from S strikes I, the index-glass, which is fastened perpendicularly to one end of the radius-bar $I V$, and movable with it; the vernier V at the other end of the radius-bar is moved along the arc $A C$ until the beam $S I$ is reflected in the direction $I H$ to strike the centre of the horizon-glass H; the beam is then again reflected in the direction $H E$, and is seen by an eye at E as if it proceeded from S' in the horizon. Produce the planes of the mirrors I and H till they meet in M; it will now be proved that the observed altitude $S E S'$ is double the angle formed by the inclination of the mirrors I and H, that is the angle $H M I$. This is the fundamental property of quadrants, sextants, Troughton's reflecting circle, and similar instruments.



PROOF. The lines in Fig. 6 correspond to those in Fig. 7, from which they are taken. Draw $I K$ and $H k$ at right

angles to I and H respectively; then by the laws of reflection $S\,I\,K = H\,I\,K$, and $I\,H\,k = E\,H\,k$.

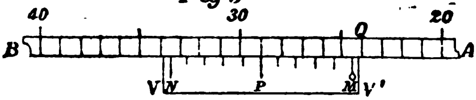
$$
\begin{aligned}
\text{Eu. I. 32:}\quad E &= S\,I\,H - I\,H\,E \\
&= 2\,(K\,I\,H - k\,H\,I) \\
&= 2\,[(90^\circ - H\,I\,M) - (90^\circ - I\,H\,k)] \\
&= 2\,(I\,H\,k - H\,I\,M) \\
&= 2\,M.
\end{aligned}
$$

That is, the observed angle $S\,E\,S'$ is double the angle formed by the inclination of the mirrors I and H.

This being the case, if the arc $A\ C$, Fig. 6, be graduated in half degrees, &c., commencing where the zero on the vernier would point when the index and horizon glasses are parallel, and every half degree on the arc be numbered as a whole one, and the subdivisions treated similarly, we shall be able to read off at sight the angle measured. Because the divisions on the arc, which in good instruments amount to 10′, are far too great for obtaining accurate results in nautical astronomy, the index at V is furnished with a scale for subdividing the divisions on the arc, and with a microscope for reading them with greater ease. The scale is called a vernier, from its inventor, Pierre Vernier, who gave a description of it in a tract published at Brussels, about A.D. 1631.

The Vernier.—Its construction may be understood from the following description. Let $A\ B$ be a portion of a straight line or circle which is divided into any number of equal parts by straight lines at right angles to it. Another scale $V\ V'$, capable of being moved evenly along $A\ B$, is the vernier.

Fig 9



If l be the length of each division on $A\ B$, and it is required to subdivide each of the divisions on $A\ B$ into n equal parts, take $M\,N = n - 1$ of the divisions on $A\ B$ and divide $M\ N$ into n equal parts by straight lines at right angles to it; then—

$$
\begin{aligned}
&\text{Length of } n \text{ divisions on the vernier} = (n-1)l \\
&\quad\text{,,}\quad\text{,,}\ \ 1 \text{ division}\quad\text{,,}\qquad\text{,,}\qquad = \frac{n-1}{n}l.
\end{aligned}
$$

If v be the length of a division on the vernier, we have

$$l - v = l - \frac{n-1}{n} l.$$
$$= \frac{l}{n};$$

or the defect of the length of a division on the vernier from the length of a division on the arc is $\frac{1}{n}$th of a division upon the arc.

As before stated, the arcs of good sextants are divided to every 10′, and the length $M\ N$ of the vernier is made equal to fifty-nine of these divisions, but is divided into 60 equal parts. Hence the difference in length between a division on the arc and a division on the vernier is from above $\frac{l}{n}$,

$$\text{i.e. } \frac{l}{n} = \frac{10'}{60} = 10'';$$

and therefore sextants thus constructed can be read off by the aid of their verniers to 10″. It is evident the length of the scale on the vernier is

$$(n-1)\ l = 59 \times 10' = 9°\ 50' \text{ of the arc,}$$

but from what has already been said, this subtends only one-half of 9° 50′, or 4° 55′ at the centre of the arc.

In practice the measurement required is that of the zero on the vernier from the zero on the limb, i.e. of M (Fig. 9) from the latter zero. It is there seen to be $24 + QM$; and $Q\ M$ is found thus:—

Let P, the pth division from M on the vernier, coincide with any division on $A\ B$, and let Q be the next less whole division on $A\ B$, then

$$QM = QP - MP$$
$$= pl - p\frac{n-1}{n} l$$
$$= pl\left(1 - \frac{n-1}{n}\right)$$
$$= p\frac{l}{n}.$$

thus $Q\ M$ is known, and hence also the distance of M from the point where the graduation begins.

Like all other delicate instruments the sextant is liable to derangement; hence the student should be able to detect when it has become so, and to place it again in perfect order. This is effected by means of what is technically known as

The ADJUSTMENTS of the sextant, which are four in number.

1. *To make the index-glass I perpendicular to the plane of the instrument.*—The index-glass is a mirror fitted into a brass frame; the whole being fastened perpendicularly to the index bar (so that it may be moved with it) by two screws directly behind the frame; farther from the frame is another screw, which is used for adjusting the glass, or setting it perpendicular to the plane of the instrument. This is done by placing the vernier near the centre of the arc, then holding the sextant by the handle with its plane upwards and arc turned away from you, look into the index glass. The portion of the arc to the right of the vernier will be seen reflected, while the arc to the left of the vernier is seen directly. If the reflected and true arcs appear as a continuous part of a circle the index-glass is in adjustment; but if the reflected arc appear lower than the true one, tighten the adjusting screw; if higher, slacken it. When once adjusted, and the screws made tight, the position of the index-glass is not very liable to derangement.

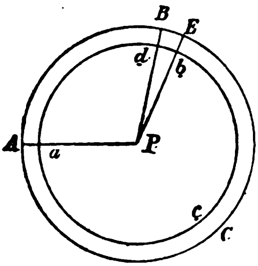
2. *To make the horizon-glass H perpendicular to the plane of the instrument.*—This is done by screwing the telescope into the collar, and looking directly at one of the heavenly bodies, a bright star being the best object that can be used; if the sun be chosen, a dark glass should be used on the end of the telescope next the eye. Move the index bar so that the image shall pass over the object, if the reflected body in its course does not exactly cover the true, the glass is out of adjustment, and is set right by its adjusting screw, which in some instruments is a milled-head one at the back of the sextant, and in others is a small screw with a hole in its top, behind and near the upper part of the glass itself, which can be turned by placing a pin in the hole.

Another method of determining whether the horizon-glass be perpendicular is: set zero on the vernier to coincide with zero on the arc, hold the instrument with its plane almost horizontally, and look through the telescope at the sea horizon, and give the instrument a slight nodding motion; if the reflected and true horizons appear in one line the adjustment is perfect; but if the image be higher the glass leans forward, and *vice versâ.* The adjustment is made as before by the screw.

3. *To make the horizon-glass H parallel to the index-glass I when the zero on the vernier coincides with the zero on the limb.*—Place the two zeros very carefully together, hold the instrument vertically, and look through the horizon-glass to the horizon. If the true horizon through the clear part of the horizon-glass appear in a straight line with the reflected in the silvered part, the two glasses are parallel; if not, they are made so by turning the lower screw at the back of the horizon-glass. Some sextants, as Troughton's pillar sextants, are not provided with means for making this adjustment, because it is not absolutely necessary; but an allowance called *index error* must be made to every observation for the want of parallelism of the two glasses when the zeros coincide.

4. *To make the axis of the telescope, when screwed into the collar, parallel to the plane of the instrument.*—Screw the inverting telescope into the collar, and, by turning the eye-tube round, bring two of the wires in its focus parallel to the plane of the instrument. Then select two heavenly bodies not less than 90° apart, and, after having made them appear of the same brightness, by means of the up-and-down screw at the back of the instrument under the collar, bring them into contact on the wire nearest the plane of the sextant: then by moving it very slightly the two bodies may be brought to the other wire; if they be still in contact the adjustment is perfect; if they separate the farther end of the telescope is inclined to the plane of the sextant. The screw in the collar farthest from the instrument must then be slackened, and the other tightened; but if they overlap the reverse is the case, and the instrument must be adjusted in a contrary manner. This adjustment is very seldom wanted at sea, because when once made perfect it is not liable to alter. The error caused by the imperfection of this adjustment is called *error of collimation*, and the observed angle is always too great. This may be shown as follows:—

Fig 10.



Let *A B C* be the great circle passing through the two points whose distance is to be measured; then, because the axis of the telescope is inclined to the plane of the sextant, therefore in sweeping for the distance it is done on a small circle

parallel to the above great circle as $a\ b\ c$; and let P be the pole of both circles. Then the angular distance $A\ a$ between the two circles is measured by the inclination of the axis to the plane of the sextant, i.e. by the error of collimation. In practice the angle $A\ P\ b$ is measured instead of the angle $A\ P\ B$, where the arc $a\ b$, or the real distance must be equal to the arc $A\ B$. Produce $P\ b$ to E, then

$$a\,b = A\,E \text{ cos. } A\,a,$$

$$\text{i.e. Cos. } A\,a = \frac{\text{arc } A\,B}{\text{arc } A\,E}$$

$$= \frac{\text{chord } A\,B}{\text{chord } A\,E}$$

$$= \frac{2 \text{ sin. } \frac{1}{2} A\,P\,B}{2 \text{ sin. } \frac{1}{2} A\,P\,E.}$$

Hence :—

Sine half the true distance $A\ P\ B$ = cos. error of collimation $\times$ sin. half the measured distance.

Ex. 21. If the line of collimation be inclined to the plane of the sextant 42′, what is the error in the observed angle 117° 22′ 50″?

Sin. half true angle = cos. error collimation $\times$ sin. half measured angle

$$= \text{cos. } 42' \times \text{sin. } 58°\ 41'\ 25''$$

42′	log. cos. =	9·999968
58° 41′ 25″	log. sin. =	9·931646
58° 41′ 0″	log. sin. =	9·931614

Hence true angle	=	117° 22′ 0″
measured angle	=	117 22 50
∴ Error in observed angle =		50 *Answer.*

INDEX ERROR.—From what has been said under the head Third Adjustment, it will have been seen, if the index and horizon-glasses be not parallel when the zeros are together, the readings will want a correction for every angle measured: this is called *index error.* It is evident if the zero on the vernier be to the left of zero on the arc when the glasses are parallel, all readings from such a sextant will be too great, and *vice versâ ;* because the graduation of the limb should begin from the point where 0° on the vernier stands when the glasses are

parallel. Owing to the unequal refraction of light through the different coloured shades, we have found a different index error for different shades; and for delicate observations the error for each should be obtained and registered.

It is determined after the adjustments have been made:—

1. By observing any well-defined *distant* object as the horizon; and moving the tangent screw until the image and object coincide. The distance the zero on the vernier is from the zero on the arc is the index error; *subtractive* if the reading be on the arc: but *additive* if the reading be off the arc.

2. By looking directly at the sun through the coloured eye-glass when the sextant is held vertically, except when the sun is very low, when it should be held horizontally; and making the sun and his image touch at their edges. The difference between the reading of the instrument and twice the semi-diameter of the sun taken for the day from the "Nautical Almanac," is the index error; *subtractive* when the reading *on* the arc is greater; *additive* when the reading *on* the arc is less, and the contrary if the reading be *off* the arc.

3. Make the sun and his image touch as in (2), then turn the tangent screw until the image passes over the object, and the two touch at their opposite edges. When the readings are one on and the other off the arc, half the difference is the index error, *additive* when the reading *off* the arc is greater, and *subtractive* when the reading *on* the arc is greater. When the readings are both on or both off the arc, half their sum is the index error to be *subtracted* when *on*, and to be *added* when *off* the arc.

As a proof of the correctness of the result obtained in (3), when one reading is on the arc and the other off, one-fourth the *sum* of the readings should be the semidiameter of the sun given in the "Nautical Almanac" for the day. But if both readings be on or both off the arc, one-fourth of their *difference* should be the semidiameter.

THE PROOF FOR INDEX ERROR.—If the zero on the vernier be to the left of zero on the arc when the index and the horizon-glasses are parallel, the reading of every observation will be too great, and the index error therefore $-$; thus, when the diameter of the sun is measured *on* the arc we get a result too great by the index error: on the contrary, if the index error be $+$ we get a result too little by that quantity. Hence

Sun's-diameter $=$ reading *on* the arc $\mp$ index error . . . (a).

If the index error be − the diameter measured *off* the arc must be too small by the index error, and too great by that quantity if the index error be +.

∴ sun's-diameter = reading *off* the arc ± index error . . . (*b*).

Take the difference between (*a*) and (*b*).

± twice index error = difference of readings *on* and *off* the arc or index error = ± ½ difference of readings *on* and *off* the arc.

Adding (*a*) and (*b*) together.

Twice diameter = reading *on* the arc + reading *off* the arc.

∴ semidiameter = ¼ [reading *on* the arc + reading *off* the arc].

Ex. 22. From the following observations find the semidiameter and the index error :—

	On the arc.		*Off the arc.*
	29′ 45″		33′ 30″
	30 0		33 30
	30 15		33 15
	90 0		100 15
	100 15		90 0
	12)190 15		6)10 15
Semidiameter	15 51·25	Index error	+ 1 42·5

On the choice of a Sextant.[1]—In choosing a sextant certain details must be attended to to ensure a good selection.

(*a*) The *joints* should be close and tight, and the *screws* act well and remain firm and steady when the instrument is shaken.

(*b*) *The graduation on both arc and vernier should be perfect.* To test this, the inlaid plates on which the graduation is made should be level with the other surfaces to which they are fixed, the vernier should be in close contact with the limb, and the divisions when viewed through the microscope should be fine and distinct. The accuracy of the graduation should be tested by bringing the zero on the vernier successively to each division on the limb, and notice every time if the last gradua-

[1] As sextants can now be tested at Kew for a small fee, no person should depend on his own judgment in the selection of one, but should demand a Kew certificate before completing the purchase.

tion on the vernier coincides with a division on the limb; if not the instrument is badly divided and should be rejected.

(*c*) *The instrument should be perfectly centred.* To examine if this be so the distance between several pairs of stars should be calculated, and afterwards measured by the sextant. If there be any discrepancy the instrument should be discarded.

(*d*) *All glasses should be perfect.* They should be of the best quality and have their faces parallel to avoid refraction, and be free from veins and streaks. To ensure this, with a small telescope look into each reflector in a very oblique direction, and observe the image of some very distant object. If the image appears single, clear, distinct, and well-defined about the edges in every part of the reflector, the glass in it is of good quality; but if it appears notched, drawn with fine lines, the glass is streaky; and if it appears double or misty about the edges the two faces of the glass are not parallel, and the instrument should be returned to the maker to have better ones inserted.

(*e*) *The shades should introduce no error.*—To test this fit the dark glass to the eye-end of the telescope, remove all shades and bring the reflected and true images of the sun into contact at their edges. Then remove the dark glass from the eye-end of the telescope, and set up first each shade separately, and afterwards their several combinations; and if, in any case, the two images do not remain in contact, the error of that shade, or combination of shades, must be registered as an error, and applied to every observation made with it.

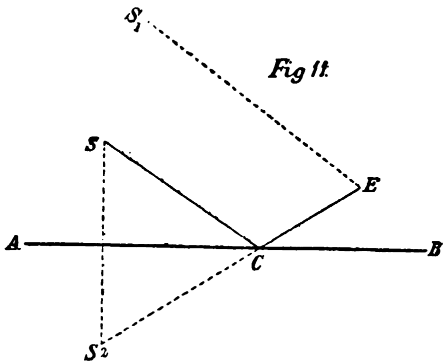


Fig 11.

The ARTIFICIAL HORIZON is a plane horizontal mirror used for taking altitudes when the sea horizon is obscured, as is often the case in high latitudes and at night. The seaman extemporizes one from a "bucket of tar," a basin of treacle, oil, or some other liquid. Whichever form is

used they all depend on the same principle in optics, viz. "The angles of incidence and reflection are equal."

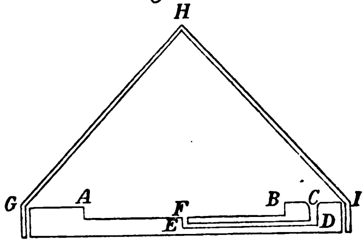
Thus if S be a celestial object, an eye placed at E sees the image S_2 as far perpendicularly below the surface of the plane horizontal mirror $A\ B$ as the object S is above it. Join $E\ S_2$ meeting the surface of $A\ B$ in C, and join also $S\ C$. Then because S is at such an immense distance from the eye compared with $E\ C$, the lines $E\ S_1$, and $C\ S$ may be considered parallel. The angle $S_1\ E\ S_2$ is measured with the sextant, while the angle $S\ C\ A$ is the altitude of the object S.

Now, the observed angle
$$\begin{aligned} S_1\ E\ S_2 &= \pi - E\ C\ S \\ &= S\ C\ A + E\ C\ B \\ &= S\ C\ A + S_2\ C\ A \\ &= \text{twice altitude of S,} \end{aligned}$$

and, therefore, when an altitude is taken with the artificial horizon, the index error of the instrument must be applied, and one-half the result must be used as the observed altitude. One great advantage arises from this method of taking observations, viz. all errors, whether of the instrument or of the observer, are reduced one-half; and because the artificial horizon is a horizontal plane no correction for dip (to be explained hereafter) is required: therefore, when practicable, the artificial is preferable to the sea horizon for taking altitudes.

One kind made by the optician consists of a plate of dark glass perfectly plane at its surface, about three or four inches in diameter, set into a brass frame, and the whole laid on three adjusting screws. When required for use the instrument is placed by the eye as nearly horizontal as can be judged, and then made perfectly so by the aid of a spirit level and the adjusting screws. From the difficulty in adjusting this kind, and the incorrect results which arise from imperfect adjustment, it has now gone almost entirely out of use.

Fig 12.



The second consists of a shallow wooden trough $A\ B$, about four inches by three, with a thick bottom and sides. It has a hole C in one of its edges communicating through a fine channel $C\ D\ E\ F$

with the bottom of the trough; into the hole C a small funnel can be screwed for the purpose of filling the trough with mercury. By this means the surface of the liquid is kept free from dross or scum (oxidation), because this being lighter floats on the surface and remains in the hole C. When the trough is nearly full it is ready for use on a calm day, but if a very gentle breeze be blowing the whole must be covered with a glass roof $G\,H\,I$, to prevent the surface being disturbed by the wind. The roof is made of a brass frame, into which are fastened two plates of glass $G\ H$, $I\ H$, whose *surfaces* are made perfectly parallel. A vertical section of the trough and roof through their centre is shown in Fig. 12. If possible, it is preferable to use this instrument without the roof; and when this cannot be done the instrument should be reversed for delicate observations to eliminate any error that may arise from imperfections in the glasses. The second will be found far preferable to the first described, because from the hydrostatic principle that "liquids maintain their level," the latter is self-adjusting. Still it possesses one great disadvantage, viz. its extreme sensibility; for, owing to the mobility of the particles of the liquid used, the least movement, shake, or tremor throws the surface into a series of undulations, and prevents accurate observations. It is this quality which renders it unfit for nautical and other purposes where a tremor may occur. We found it quite impossible to take observations with the artificial horizon from the top of the Navigation School by night (by day we have the sea horizon), owing to the shaking of the walls of the building by wind or passing vehicles. At the suggestion of the late Commander Walker, R.N., we had a horizon constructed, so that a piece of glass, whose surfaces were perfectly plane and parallel to one another, should float on the mercury, and fit so close to the sides of the trough as to prevent any great motion, yet not so close as to prevent its free action. At first we found very great discrepancies, owing to the glass not being homogeneous, and thus floating slightly deeper at one end than at the other; but the idea occurred to us of taking two sights with the instrument reversed, using the means of the altitudes and of the times of observation. Thus the error arising from the want of homogeneity was eliminated, and we have since found this a very efficient instrument. If the observer pleases he can obtain the index error of the instrument for each edge of the glass nearest to him

and then one sight will suffice; but it will, when possible, be preferable to take two sights with the instrument reversed. It has occurred to us that such a horizon mounted in gimbals would prove a valuable instrument for sea purposes. After using the one made for us for about two years, we were gratified in seeing the same method recommended by the celebrated astronomical instrument maker, Mr. Cooke, of York.[2]

Another disadvantage arises from using the artificial horizon when a sextant is employed: viz. that no altitude of an object above 60° can be observed, because the sextant is graduated only about 120°; but if Troughton's reflecting circle be used then any altitude can be measured by this horizon.

Attempts have been made to fit an artificial horizon to sextants; but we have never yet seen one susceptible of any great degree of accuracy.

THE CHRONOMETER.—The chronometer is a superior kind of watch, so constructed that its daily loss or gain through variations of temperature are reduced to a minimum. This is effected by using a compensation or expansion balance, which is formed of a combination of metals with different co-efficients of expansion, such as brass and steel. Like the common watch the moving power is supplied by the main-spring, but the tapering drum or fusee on which the chain turns when the chronometer is wound up is of the shape of a hyperbola. Thus it acts like a lever of variable length, and causes the force exerted by the main-spring to remain constant, and thus helps to preserve the isochronism of the chronometer, which is the great desideratum.

The machinery is of such delicate construction, that the greatest possible care must be taken of it both at sea and in harbour. All jerks, vibrations, and twisting of the chronometer should be avoided. It should be wound up *every morning* at the same hour, the key being placed firmly in the hole and turned steadily through each half-turn. On no account should it be moved from its place on board, which should be as near as possible to a horizontal line passing through the centre of gravity of the vessel, because on that line there is the least oscillatory movement in the vessel. When observations are made, the time of each sight is noted by a good watch, which must be compared with the chronometer both before and after the sights are taken; and the time by chronometer at the instant

[2] Monthly Notices, Royal Astronomical Society, April, 1865.

of taking the sights is thence deduced from the time by watch.

Chronometers are slung in gimbals to keep the face horizontal, because a change in position through the heeling of the ship would cause a change in their rates. The box which holds a chronometer is fitted snugly into a well-padded outer case to prevent sudden jerks or jars; and the case is screwed to a fixed support, which is usually constructed for its reception; in large ships a room is set apart for the purpose. An extensive series of experiments has been conducted from time to time, with the view of finding out the best place in a vessel to keep the chronometer; the best method of stowing it, whether by suspension or otherwise; the effects of a change in temperature; and of the magnetism acquired by the iron used in the construction of the vessel, &c., &c. For a full description of these the student is referred to Captain Shadwell's "Notes on the Management of Chronometers."

ERROR AND RATE OF CHRONOMETERS.—By Englishmen longitude is reckoned from Greenwich, and as this is the difference between local mean time and that at Greenwich, some means are necessary for finding Greenwich mean time. For this purpose a chronometer is used, and before being taken to sea its error on Greenwich mean time and its rate are ascertained and supplied to the officer in command.

The error of a chronometer for any place is the difference between the time shown by it and the correct mean time at that place; and is said to be *fast* when the chronometer shows a later time, and to be *slow* when it shows an earlier time than that at the place.

The rate of a chronometer is the daily change of its error, and is said to be *gaining* when the error is fast and increasing or slow and decreasing, and to be *losing* when the error is fast and decreasing, or slow and increasing.

By these means the mean time at Greenwich can always be found, and this is necessary in order to correct much of the data taken from the "Nautical Almanac," and also to find longitude.

TO FIND GREENWICH MEAN TIME BY CHRONOMETER.—1. Express the time by chronometer astronomically, and add the original error if the chronometer were slow when the error was found; but subtract if it were fast.

2. Find the number of days elapsed since the chronometer was

rated, and multiply this by the daily rate, the product is the accumulated rate, which must be added if the chronometer be losing, but subtracted if gaining.

Ex. 23. Find the mean time at Greenwich if the chronometer showed 10 hrs. 41 m. 21 secs. a.m. on September 29th. It had been found to be 47 m. 35 secs. fast on June 13th, and was losing 2·3 secs. daily.

Here the time must first be expressed astronomically; that is, if it be a.m. twelve hours must be added to the time and the date placed the day before; but for p.m. astronomical and civil time are the same. In this case the astronomical time is September 28th, 22 hrs. 41 m. 21 secs.; then

	Hrs.	min.	sec.		
Astronomical time, Sept. 28	22	41	21	Elapsed days	107·9
Fast June 13	–	47	35	Daily rate	2.3
	21	53	46		3237
Accumulated loss	+	4	8		2158
Greenwich mean time, Sept. 28	21	57	54		60)24,8·17
				Total loss	4m. 8s.

EXERCISE II.

Ex. 24. State the principle of the sextant. How large an angle can I measure with a sextant whose limb contains 70°? What do you mean by the index error? Show how to find it from the following observations, and also the sun's semidiameter:—

On the arc.	*Off the arc.*	
36′ 40″	26′ 30″	
36 40	26 40	
36 20	26 50	*E.* 1869.

Ex. 25. What causes the sextant to be so useful an instrument to the navigator? Explain what the index error is, and describe a method of determining it. From the following observations find the sun's semidiameter and the index error of the sextant.

On the arc.	*On the arc.*	
7′ 20″	1° 10′ 10″	
7 10	1 10 10	
7 20	1 10 10	*E.* 1877.

Ex. 26. What are the adjustments of a sextant? Show how

to find the index error by measuring the diameter of the sun *on* and *off* the arc.

The reading *on* the arc is 31′ 30″, and that *off* the arc is 33′ 40″. What is the amount and algebraical sign of the index error? *E.* 1871.

Ex. 27. Describe the several methods of finding the index error of a sextant; can it be proved approximately that the real error has been found. *Royal Naval College*, 1867.

Ex. 28. Prove the principle on which the division of the limb of the sextant is made. How would you ascertain whether the horizon-glass is perpendicular to the plane of the instrument? In determining the index error by making the reflected sun touch the true sun on each side of zero, the reading on the off-side is 32′ 50″, and on the on-side is 32′ 20″; what is the amount and sign of the index error? What is the sun's semidiameter for the day on which the observation is made? *E.* 1872.

Ex. 29. Explain the method of finding and correcting an error of parallelism in a fixed reflector of a sextant.
Royal Naval College, 1867.

Ex. 30. In observing with a sextant after having arranged your shades you find that the reflected sun is not bright enough compared with the direct sun; how would you correct this? Explain how you test the adjustments of the index-glass. *E.* 1873.

Ex. 31. Describe all the adjustments of the sextant, showing how to detect any want of adjustment in each case. *A.* 1875.

Ex. 32. Describe fully the sextant, drawing a figure of one to indicate the different essential parts. What is the real length of the limb which is graduated from 0° to 28°. *E.* 1880.

Ex. 33. State how the limb of the sextant is graduated, and prove that the angle between the direction of a ray when it strikes the index-glass, and its direction when reflected from the horizon-glass is equal to twice the inclination of these two mirrors. How can you discover whether the line of collimation is in adjustment or not? *A.* 1880.

Ex. 34. Explain clearly the principle of the vernier, and show what arrangements of it are necessary in instruments for measuring angles to enable us to read off on the vernier to the same number of *seconds* as the limb is graduated to *minutes*.
A. 1869.

Ex. 35. Mention all the instruments to which you know the vernier is attached, and explain how they are divided.
Royal Naval College, 1873.

Ex. 36. Prove that the altitude of a body observed by means of an artificial horizon is double the real altitude.

Royal Naval College, 1873.

Ex. 37. Explain the construction and use of the artificial horizon. Why is there no *dip* to be allowed for? Which are the angles of incidence and reflection?

Royal Naval College, 1872.

Ex. 38. What is a chronometer? Describe how a chronometer enables you to determine your longitude; and show the importance of knowing accurately its "*error*" and "*rate*." Define these terms. *E.* 1878.

Ex. 39. What is the line of collimation of the sextant? State when it is in adjustment: show how you find whether it is so or not, and describe the method of adjusting.

An angle is observed 126° 30′ 10″, when the line of collimation is inclined 30′ to the plane of the sextant; find the error in the angle observed. *Honours*, 1873.

Ex. 40. Investigate formulæ for computing the several corrections of the sextant, such as the index correction, and error in the line of collimation. The error of collimation in a sextant is 1° 10′; required the consequent error in an angle of 100°.

For Beaufort Testimonial, 1865

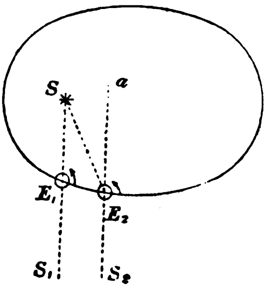
CHAPTER IV.

Time—Sidereal day and year—Solar day and year—Precession of the equinoxes—Unit for measuring intervals of time—To convert intervals of solar time into equivalent intervals of sidereal time, and *vice versâ*—How decimal multipliers are obtained for that purpose—Time and longitude—Civil and astronomical time—Greenwich date—Ambiguity of date obtained from time by chronometer—Exercise—Examination.

No distinct notion of time can be obtained without referring it to a succession of events or phenomena; and the most convenient phenomena to which we can refer it are those which are universal, constant, and isochronous in their character, and hence can be used by every person on the surface of the earth for the purpose intended. Such are those supplied by the rotation of the earth on her axis, thus defining the *day;* and the revolution of the earth around the sun, thus giving the *year.* The interval taken up by one complete rotation of the earth on her axis is called a *sidereal day;* and the interval elapsed in one complete revolution of the earth in her orbit, measured by the time between leaving a fixed point in the heavens and returning to it again, is called a *sidereal year.* These then are the natural units of time. For everyday purposes these are difficult of observation, and therefore events more easily observed are chosen; and as the apparent motion of the sun in the heavens is one of the most striking phenomena with which mankind is acquainted, we have recourse to his motions for defining one unit. The *solar day* is the interval elapsed between two successive transits of the sun across any given meridian; and the *solar* or *tropical year* is the interval between two successive passages of the sun through the "first point of Aries." If this point were fixed, the sidereal and solar years would be of the same length; but owing to the action of the moon and sun on that portion of the earth near the equator which bulges beyond the inscribed sphere, and gives to it its spheroidal shape, the "first point of Aries" moves backward on the ecliptic at the rate of $50''\cdot224$ per annum. This is called the *precession of*

the equinoxes, and causes the sidereal year to be longer than the tropical year by the time it takes the sun to describe the arc 50″·224. On the other hand, the sidereal day is shorter than the solar day, because, whilst the earth is rotating on her axis, she is also moving on in her orbit, and hence, when a star is observed on the same meridian as it was on the preceding day, the sun has not yet arrived at the meridian. This will be better understood from the following figure, which for the sake of clearness is greatly exaggerated. Let S be the sun, E_1 E_2 the earth in her orbit, S_1 S_2 the same star, which is so distant that lines drawn to it from any part of the earth's orbit are parallel. Now when the earth is at E_1, let the sun and star be on the meridians 180° apart at the same instant, then the earth rotates on her axis from west to east, and moves on in her orbit from E_1 to E_2. Let E_2 be the position of the earth when the star S_1 next comes into the meridian at S_2: it is evident from the figure that the sun has not yet arrived at its former meridian, but the earth must rotate through an angle a E_2 S before that event can occur. This is the reason why the sidereal day is shorter than the solar day. Now this continues throughout the whole year, and thus it is seen that the earth rotates on her axis once more in a solar year than the number of days in that time, and therefore, roughly speaking, the length of a solar day is to the length of a sidereal day as 366 : 365.

Fig 13.



For nautical purposes the unit of time employed is the day, which may generally be defined as the interval elapsed between two successive transits of a heavenly body. The interval between two successive transits of the same star is a *sidereal day;* that between the transits of the sun a *solar day;* and that between the transits of the moon a *lunar day.* Each day is supposed divided into twenty-four equal parts, called hours. The two latter days are of variable length; but for astronomical purposes an interval of time which shall be invariable is necessary as a unit; this is supplied by the sidereal day.

A SIDEREAL DAY is the interval of time between two successive transits of the "first point of Aries," and begins at the instant

of transit. The first point of Aries is not a fixed point in the heavens, hence the interval between two successive transits of a star is not the same as a sidereal day; but the motion of that point is so small (only $50''\cdot224$ annually) that no practical inconvenience is experienced in Nautical Astronomy by assuming the sidereal day to be the same as the interval between two successive transits of the same star across any given meridian. But, owing to the annual revolution of the earth, an inconvenience is experienced by selecting this unit, because the transit of the first point of Aries takes place sometimes during our waking hours, and at other times during the time allotted to sleep; and thus we have the sidereal day beginning at various times in darkness and in light. Recourse has therefore been had to another day for civil matters, whilst the sidereal day is almost wholly used for calculations in the observatory. As the time the sun is above the horizon governs the social life of mankind, his transit has been adopted as the beginning of the day.

The APPARENT SOLAR DAY is the interval between two successive transits of the sun's centre across the same meridian, and begins when the sun's centre is on the meridian. This is called *noon*. Now the sun completes his apparent revolution among the stars in a year, hence his difference in right ascension must average nearly $1°$ per day, and the earth will thus have to turn nearly $361°$ on her axis to complete a solar day, as has already been shown, and hence the solar day, as will be proved hereafter, must be nearly four minutes longer than a sidereal day.

In the adoption of the solar day as the unit we are met with the difficulty that it is variable in length. This arises from two causes: *first*, the earth has an unequal motion in her orbit, or what amounts to the same thing, the sun has an apparent unequal motion in the ecliptic varying from the 1st of July, when he sweeps out an angle of $0° 57' 11''\cdot5$ per day, to the 31st of December, when the angle is increased to $1° 1' 9''\cdot9$ per day. *Secondly*, even if his motion were uniform in the ecliptic, his motion in right ascension would vary from the fact that the equinoctial, which is at right angles to the axis, makes an angle with the ecliptic, and hence his motion in right ascension varies from the time when he is moving parallel to the equinoctial, as he must do at his maximum declination, to the time when he crosses the equinoctial and his path makes an angle with it called the obliquity of the ecliptic. In the former case his motion in right ascension must be quicker

than in the latter; from both these causes the solar day must vary in its length. But a day varying in length, that is, a constantly varying unit of duration, would be most inconvenient in practice, and hence the following method is used to obviate this. An imaginary body, called the mean sun, is supposed to move along the equinoctial at the same mean angular velocity as the true sun in the ecliptic, and as the motion is constant, the days as marked out by the mean sun will be equal in length, and will be the average of all the apparent solar days in the year. This is the mean solar day.

A MEAN SOLAR DAY is the interval of time between two successive transits of the mean sun across any meridian. It begins when the mean sun is on the meridian. Next suppose a star to start with the true sun when in perigee, and to travel in the ecliptic with the sun's mean velocity. Here the sun is travelling faster than in any other part of his orbit, and will soon outstrip the star; but as they proceed the sun's motion will get less and less, and by-and-by the star will have caught up the sun. This will happen when the sun is in apogee, and is then travelling slower than in any other part of his orbit. The star having the mean motion of the sun will then outstrip him, and as the sun's motion increases, he will again catch up the star in perigee, the two bodies being always together in perigee and apogee. Now let the starting-point of the star be so adjusted that it may be at the first point of Aries at the same time as the mean sun, then as the star and mean sun have the same velocity, the latter will describe an arc of the equinoctial in the same time as the star describes an equal arc of the ecliptic, and thus the right ascension of the mean sun will be the same as the longitude of the star. Thus the connection of the two suns (the true and the mean) is established, and may be expressed by saying that the right ascension of the mean sun is equal to the mean longitude of the true sun. From this consideration it is evident that the real sun in the ecliptic and the mean sun in the equinoctial are always near the same hour circle, and will transit at times not far removed from each other. The interval between the transits is called the *equation of time.*

For comparing the lengths of the years and days, the passages of the sun through the first point of Aries at very distant epochs have been carefully observed, and the interval thus found, divided by the number of revolutions of the earth in her orbit, have given the length of the tropical year. The best authorities

agree that the length of this year is 365·242216 mean solar days, or 365 days 5 hrs. 48 mins. 47·808 secs.; but, as has been shown, because the earth turns once more on her axis than the number of solar days in the year, the length of the solar year is 366·242216 sidereal days; hence

$$365{\cdot}242216 \text{ mean solar days} = 366{\cdot}242216 \text{ sidereal days.}$$

$$\therefore \text{ a mean solar day} = \frac{366{\cdot}242216}{365{\cdot}242216} \text{ sidereal days}$$

$$= \text{a sid. day} + \frac{1}{365{\cdot}242216} \text{ sid. day}$$

$$= \text{a sid. day} + 3\text{m. } 56{\cdot}555\text{s. sid. time}$$

$$= 24 \text{ hrs. } 3\text{m. } 56{\cdot}555\text{s. of sidereal time.}$$

Also

$$\text{A sidereal day} = \frac{365{\cdot}242216}{366{\cdot}242216} \text{ mean solar day}$$

$$= \text{a mean solar day} - \frac{1}{366{\cdot}242216} \text{ mean solar day}$$

$$= 24 \text{ hrs.} - 3\text{m. } 55{\cdot}909\text{s. mean solar time}$$

$$= 23 \text{ hrs. } 56\text{m. } 4{\cdot}09\text{s. mean solar time.}$$

Sidereal intervals and sidereal time have often in Astronomy to be converted into mean solar intervals and time, and *vice versâ*. For this purpose tables called "Tables of time equivalents" are inserted in the "Nautical Almanac," at pages 480 to 483, for the year 1887. They are easily computed thus:—A solar year has been shown to consist of 365·242261 mean solar days, and also of 366·242261 sidereal days; hence

$$\text{Mean solar interval} : \text{sid. interval} :: 365{\cdot}242216 : 366{\cdot}242216;$$

$$\therefore \text{ sid. interval} = \frac{366{\cdot}242216}{365{\cdot}242216} \text{ of mean solar interval}$$

$$= 1{\cdot}0027379 + \ldots \text{ of mean solar time.}$$

It is used thus:—

Ex. 41. Reduce 7h. 40m. 20s. of mean solar time to its equivalent interval in sidereal time.

7h. M. solar time	= 7 × 1·0027379 =	7h.	1m.	8·995s.	sid. time.
40m. ,,	= 40 × 1·0027379 =		40	6·571	,,
20s. ,,	= 20 × 1·0027379 =			20·055	,,
∴ 7h. 40m. 20s. M. solar time =		7h.	41m.	35·621s.	sidereal time.

And the sidereal time required corresponding to the mean solar

time on any particular day = sidereal time at the preceding noon + the equivalent to the given mean solar time.

From the same proportion above—

$$\text{Mean solar interval} = \frac{365{\cdot}242216}{366{\cdot}242216} \text{ of sidereal time}$$
$$= {\cdot}9972696 + \ldots \text{ sidereal time.}$$

Ex. 42. Reduce 7h. 40m. 20s. of sidereal time to its equivalent interval in mean solar time.

7h. sid. time	= 7 × ·9972696 =	6h.	58m.	51·194s.	M. solar time.
40m. „	= 40 × ·9972696 =		39	53·447	„
20s. „	= 20 × ·9972696 =			19·945	„

∴ 7h. 40m. 20s. sid. time = 7h. 39m. 4·586s. of mean solar time.

And the mean solar time required corresponding to the sidereal time on any particular day = mean time at the preceding sidereal noon + the equivalent to the given sidereal time.

On working these examples with the "Tables of time equivalents" in the "Nautical Almanac" they will be found to agree with the above results obtained by using the decimal multipliers.

Time and Longitude.—It will be often necessary to convert longitude into time, and time into longitude, in order to find the Greenwich time from local or ship time, and *vice versâ*. The unit, the mean solar day, is divided into hours, minutes, and seconds of mean solar time; but the student will have seen that degrees are also divided into minutes and seconds; and hence we have the same terms minutes and seconds used to denote different things, viz. intervals of duration and of space. To make a distinction and to prevent confusion, minutes and seconds of time are marked by the initial letters "m." and "s.," whilst minutes and seconds of arc are marked (′) for minutes and (″) for seconds. Hence if we write 17 minutes 43 seconds of time it is 17m. 43s.; but 17 minutes 43 seconds of angular measurement are written 17′ 43″; and this notation must be rigorously followed by the student.

The earth turns on its axis in 24hrs. mean solar time,

i.e. 360° of longitude are equivalent to 24hrs.

$1°$ „ is „ $\frac{24}{360}$hrs. = 4m.

$1'$ „ „ „ $\frac{4}{60}$m. = 4s.;

therefore a difference of 4 minutes in time between two places corresponds to a difference of a degree in longitude between the places; and 4 seconds of time correspond to a minute of longitude. Hence to convert longitude into time, all that is necessary is to multiply the degrees, minutes, &c., of longitude by *four*, and the result is minutes and seconds of time.

Ex. 43. What difference in time is equivalent to 87° 24′ 30″ difference of longitude?

$$\begin{array}{r} 87^\circ\ 24'\ 30'' \\ 4 \\ \hline 60)349\ \ 38\ \ 0 \\ \hline 5\text{h. } 49\text{m. } 38\text{s.} \end{array} \quad \textit{Answer.}$$

Similarly, to convert intervals of time into difference of longitude, we must reduce the time to minutes and seconds and divide by *four;* the quotient is the difference of longitude expressed in degrees, minutes, and seconds.

Ex. 44. What difference of longitude corresponds to 5h. 49m. 38s. of mean solar time?

$$\begin{array}{l} 5\text{h. } 49\text{m. } 38\text{s.} \\ 60 \\ \hline 4)349\ \ 38\ \ 0 \\ \hline \ \ 87^\circ\ 24'\ 30'' \end{array} \quad \textit{Answer.}$$

CIVIL AND ASTRONOMICAL TIME.—For civil purposes it is convenient to have the day begin at midnight, because then the day of the month has not to be changed during business hours; but for astronomical purposes the beginning of the day is marked by the sun's upper transit across the meridian; hence the same absolute moment may be one day in civil reckoning and the day before in astronomical reckoning. This occurs for all times between midnight and the following noon. It has been agreed that the astronomical day shall begin at noon and the hours counted through the whole twenty-four, whilst the civil day shall begin at the preceding midnight and be counted in two intervals of twelve hours each; before noon being marked a.m., and after noon marked p.m. of the same day. The civil day is therefore twelve hours old when the astronomical day begins.

Thus if an event happened at half-past eleven in the morning of the 24th of August it would be marked thus:—

In astronomical time, 23d. 23h. 30m.
In civil time, 24d. 11h. 30m. a.m.

Hence we deduce the rule: If a.m. civil time be given to find astronomical time, add 12 hours to it and date the day one before the civil date; and astronomical time may be reduced to civil time by rejecting 12 hours when they are above that number and dating the day one later, calling the time a.m. of that day. Under other circumstances the two agree in date.

GREENWICH DATE.—It is evident that as the earth rotates on her axis from west to east, a place to the eastward of another must have sunrise, noon, and sunset before the other place; hence at the former place the time of day is the later of the two. Thus for the same day of the month it is later in the day at Calcutta than at Bombay, at Bombay than at Suez, at Suez than at Plymouth, and so on, the places farthest to the westward have, the earlier time for the same absolute moment. From the above considerations we deduce, that to find Greenwich time from local or ship time, convert the longitude into its equivalent time and *add* to the ship time if the longitude be *west*; but *subtract* it from the ship time if the longitude be *east*. The reverse of this holds good if Greenwich time be given to find local or ship time.

Ex. 45. If it be June 11d. 4h. 29m. 13s. p.m. at Calcutta in longitude 88° 24′ 56″ E., what is the time at Greenwich?

Long. 88° 24′ 56″ E.	Civil time at Calcutta, June	11d.	4h.	29m.	13s.
4	Long. in time, E. ∴ −		5	53	39·7
60)353 39 44					
Long. in time 5h. 53m. 39·7s.	Civil time, Greenwich, June	11	10	35	33·3 a.m.

Ex. 46. If the astronomical time at Greenwich be July 5d. 2h. 39m. 53s., what is the civil time at Valparaiso in longitude 71° 41′ 30″ W.?

Long. 71° 41′ 30″ W.	Greenwich astronomical time, July	5d.	2h.	39m.	53s.
4	Long. in time, W. ∴ −		4	46	46
60)286 46 0					
Long. in time 4h. 46m. 46s.	Valparaiso astronomical time, July	4	21	53	7
	Valparaiso, civil time, July	5	9	53	7 a.m.

Ex. 47. An observation was made at New York in longitude 74° 2′ 30″ W. on October 1st, 9h. 25m. 16s. a.m.; what was the corresponding astronomical time at Sydney in longitude 151° 14′ 30″ E.?

Long., New York,	74° 2′ 30″ W.	
Long., Sydney	151 14 30 E.	
Diff. long.	225 17 0 E.	
	4	
	60)901 8 0	
	15h. 1m. 8s.	

	d.	h.	m.	s.
Civil time, New York, Oct.	1d.	9h.	25m.	16s. a.m.
		12		
Astron. time, New York, Sept.	30	21	25	16
Long. in time, E. ∴ +		15	1	8
Astron. time, Sydney, Oct.	1	12	26	24

AMBIGUITY OF TIME BY CHRONOMETER.—As chronometers, like watches and clocks, show only 12 hours, an ambiguity exists concerning the date at Greenwich, that is, whether the hours, minutes, and seconds should be reckoned as so many after noon or after midnight. This uncertainty is removed by applying the longitude in time from the dead reckoning to the time at ship: thus—

Ex. 48. Suppose a chronometer shows 7h. 25m. 15s. when the time at ship is 25th February, 3h. 16m. 25s., a.m., and longitude about 62° 15′ W.; find the Greenwich astronomical time.

	d.	h.	m.	s.
Time at ship	24d.	15h.	16m.	25s.
Long. in time, W. +		4	9	0
Greenwich date, nearly	24	19	25	25

Long =	62° 15′ W.
	4
	60)249 0
Long. in time	4h. 9m. 0s.

Here we see the 7 hrs. shown by the chronometer is 7 hrs. after midnight, and should be interpreted 19 hrs. The Greenwich time is therefore February 24d. 19h. 25m. 15s.; the few seconds' difference between the time at ship and that by chronometer being caused by the uncertainty in the longitude.

Ex. 49. 1887, on 15th of April, about 2 p.m. at ship, a chronometer showed 5h. 0m. 28s., and had been found 1m. 13s. fast of Greenwich mean time on the 9th of January, and to be correct on the 3rd of March; what was its rate, and the true mean time at Greenwich if the ship was in longitude 133° 41′ E.?

For G. M. time nearly.					*Long. in time.*
Time at ship, April	15d.	2h.	0m.	0s.	Long. 133° 41′ E.
Long. in time, E. ∴ −		8	54	44	4
G. M. time nearly	14	17	5	16	60)534 44
					8h. 54m. 44s.

From the Greenwich mean time thus found we see that the 5 hours shown by the chronometer was in the morning, and should therefore be interpreted April 14 days 17 hrs. 0 min. 28 secs.

To find the rate.			*To find accumulated error.*	
Chro. fast	1m. 13s.	on Jan. 9.	Elapsed time	42·71 days.
„ „	0 0	on Mar. 3.	Rate losing	1·38
Loss in 53 days	1 13			34168
„ 1 day	1·38s.			12813
				4271
			Total loss	58·9398s.

For Greenwich mean time.

Time by chronometer, April		14d.	17h.	0m.	28s.
Loss in 42·71 days from March 3 to April 14	+				59
∴ Greenwich mean time, April 14		14	17	1	27

EXERCISE III.

Ex. 50. Define mean solar time, apparent time: show how you convert one into the other. Explain fully the causes of the difference between mean and apparent time. What do you mean by sidereal time? *E.* 1873.

Ex. 51. Define the term "mean solar year;" find its exact length. *Royal Naval College*, 1867.

Ex. 52. What is the civil time corresponding to the astronomical time, April 5d. 22h. 6m.? What is the difference in time between Constantinople (long. 28° 59′ 15″ E.) and Funchal (long. 16° 54′ 45″ W.)? Explain clearly the reason of this. *E.* 1877.

Ex. 53. Define hour angle of a body, apparent solar day, mean solar day. How are apparent and mean solar times converted

into one another? Two ships leave the Thames, off Greenwich, at the same time, one sailing west round Cape Horn, the other east round the Cape of Good Hope, and they meet in longitude 180°; what will be their difference in time? *E.* 1879.

Ex. 54. Define the term "day." Show clearly why the apparent solar day is variable in length.
Royal Naval College, 1874.

Ex. 55. What is the difference of time between Constantinople (long. 28° 59′ 15″ E.) and Calcutta (long. 88° 27′ E.)? What is the difference in longitude between two places whose difference of time is 2h. 40m. 15s.? Explain how the difference of time is used to determine the longitude, mentioning how the time at the two places is severally found. *E.* 1879.

Ex. 56. At ten o'clock on this day at Greenwich, in what longitude is it midnight? *B.A. London*, 1839.

Ex. 57. If 24 mean solar hours = 24h. 3m. 56·555s. sidereal hours, express 2d. 10h. mean solar time in sidereal time.
Royal Naval College, 1865.

Ex. 58. What is the distinction between mean and apparent time? Which is used on board ship at sea? Define "Greenwich date." What is the difference in time at Cairo and Ascension, the longitude of the former being 31° 19′ E., and that of the latter 13° 59′ W. Explain clearly the method by which you solve this question. *E.* 1880.

Ex. 59. Explain the apparent motion of the sun through the fixed stars, and the consequent difference between mean solar time and sidereal time. *B.A. London*, 1840.

Ex. 60. Define a mean solar year and a sidereal year. Which is the longer, and why? How was the length of a mean solar year determined? *Royal Naval College*, 1864.

Ex. 61. The length of the mean solar year being 365·242264 days, find the daily motion of the mean sun in the equinoctial.
Royal Naval College, 1864.

Ex. 62. How is sidereal time converted into mean solar time, and the converse? *For Beaufort Testimonial*, 1864.

Ex. 63. In a mean solar day a meridian revolves through 360° 59′ 8·33″. Explain this. Why is a sidereal day shorter than a mean solar day, but a sidereal year longer than a mean solar year? *Royal Naval College*, 1865.

Ex. 64. Given the length of a sidereal day (24 hours), to find its length in solar time. *Royal Naval College*, 1869.

Ex. 65. Explain the term "mean sun," and assuming the

length of the mean solar year, find the amount of the mean sun's daily motion in the equinoctial, and give its direction.

Royal Naval College, 1869.

Ex. 66. Find the arc described by a meridian of the earth in a mean solar day; give a figure showing its direction.

Royal Naval College, 1872.

Ex. 67. Distinguish a solar day, a mean solar day, and a sidereal day in mean solar hours and minutes. Distinguish a "tropical" and a "sidereal" year.

B.A. and B.Sc. London, 1880.

Ex. 68. Convert 13h. 18m. 25s. mean solar interval into a sidereal interval by the use of the decimal multiplier, and prove the result by the table of "time equivalents."

Ex. 69. Find the equivalent in mean solar interval of 20h. 12m. 34s. of sidereal interval by both methods.

Ex. 70. Reduce 18h. 9m. 10s. of mean solar interval to a sidereal interval by using both methods.

Ex. 71. Reduce 18h. 9m. 10s. of interval in sidereal time to its equivalent in mean solar time, and show that both methods agree.

Ex. 72. Express the following longitudes in their corresponding time equivalents :—

(*a*)	24°	21′	20″
(*b*)	10	56	30
(*c*)	170	18	15
(*d*)	41	40	24
(*e*)	89	39	50
(*f*)	100	25	18

Ex. 73. What differences in longitude are equivalent to the following differences of time ?—

(*a*)	6hrs.	8m.	21s.
(*b*)	12	15	14
(*c*)	0	27	30
(*d*)	9	12	18
(*e*)	0	0	18
(*f*)	15	53	27

Ex. 74. If it be half an hour before noon on March 28th at Plymouth, in longitude 4° 7′ 16″ W., what is the astronomical date at Rangoon, in longitude 96° 13′ 10″ E. ?

Ex. 75. Convert 13h. 18m. 25s. time into longitude. What time will it be at Greenwich when it is noon at Calcutta, longitude 88° 24′ 56″ E. ?

Ex. 76. If it be March 1st, 2h. 58m. 53s. at Sydney, in longitude 151° 14′ 30″ E., what is the time at Greenwich?

Ex. 77. If it be noon at Valparaiso, in longitude 71° 41′ 30″ W., what is the time at Greenwich, and also at Sydney, longitude 151° 14′ 30″ E.?

Ex. 78. Find the Greenwich time, if a chronometer showed March 16th day, 4h. 15m. 44s., the time at ship in longitude about 132° W. being 8h. 28m. a.m. On January 1st it was slow 1h. 1m., and on February 5th it was slow 1h. 0m. 42s.

Ex. 79. If on September 1st a chronometer showed 11h. 2m. 17s. when the time at ship in longitude 60° 5′ W. was about 7h. 13m. 16s. a.m., find the Greenwich mean time if on June 30th the chronometer showed Greenwich mean time, and on the 27th of July it was slow 4m. 50s.

Ex. 80. Before going to sea a chronometer was rated on the 8th of January, and found to be fast 2h. 52m. 22s., and again on the 3rd of February was found to be fast 2h. 56m. 50s. Find its rate, and if it showed noon on the 15th of April find the correct Greenwich mean time.

Ex. 81. On the 27th of December, 1886, a chronometer was found to be fast of Greenwich mean time 2m. 28s., and on the 25th of February, 1887, it was found to be slow 7m. 44s.; find its rate. If the ship be in 180° West longitude on October 15th, about 9 a.m., and the chronometer showed 8h. 41m. 59s., find the Greenwich mean time.

CHAPTER V.

Corrections for observed altitudes—Index error—Dip—Proof of formula—Semidiameter—Augmentation of semidiameter—Proof of formula—Refraction—Laws of refraction—Bessel's and Bradley's formula—How modified for pressure and temperature—Uncertainty of these laws—Effects of refraction on the diameter of an object—Parallax—Proof of formulæ employed—Reduction of moon's horizontal parallax—Distances and sizes of objects calculated from their parallaxes—Reduced latitude and angle of the vertical—The same at any azimuth—How diurnal parallax is found from observations on the same meridian—Exercise, examination.

THE ALTITUDE of a heavenly body is its angular distance above the horizon measured on a vertical circle.

The OBSERVED ALTITUDE is the altitude above the visible horizon.

The APPARENT ALTITUDE is the altitude above the sensible horizon.

The TRUE ALTITUDE is the altitude above the rational horizon.

The corrections necessary to reduce the observed to the apparent altitude are:—

1. That due to the error of the instrument, called *index error.*

2. That due to the elevation of the observer above the surface of the earth, called *dip,* or *depression of the horizon.*

3. That due to the size of the disc of the object observed, called *correction for semidiameter.*

To reduce the apparent to the true altitude two other corrections are necessary, viz. :—

1. That due to the effects of the atmosphere, called *correction for refraction.*

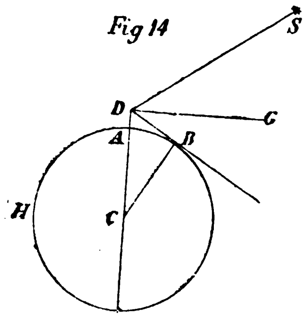
2. That due to the position of the observer on the surface of the earth instead of at its centre. This correction is called that for *parallax.*

1. INDEX ERROR is that caused by the want of parallelism between the index and horizon glasses of a sextant, and has already been discussed when treating of that instrument.

2. DIP, OR DEPRESSION OF THE HORIZON, is the angle at

the observer's eye between the visible and sensible horizons.

Fig 14



Let S be a heavenly body, $A\ B\ H$ a section of the earth through its centre C, whose plane produced would pass through S, $A\ D$ the height of the observer. Draw $D\ A\ C$ to the centre of the earth, and $D\ B$ touching the great circle in B, join $C\ B$. Draw $D\ G$ at right angles to $D\ C$, then $D\ G$ is parallel to both the sensible and rational horizons, $D\ B$ is a line drawn to the visible horizon, and (Eu. III. 18) the angle at B is a right angle. In practice the observer brings the reflected image of the object S in a line with the visible horizon, that is, he measures the angle $S\ D\ B$, and calls it *the observed altitude.* It is evident the angle $S\ D\ B$ is greater than the angle $S\ D\ G$, the apparent altitude by the angle $G\ D\ B$, which is called the *dip.*

Now $$S\,D\,G = S\,D\,B - G\,D\,B,$$
or, $$\text{App. alt.} = \text{observed alt.} - \text{dip.}$$
Hence this correction is always *subtractive.*

And because $C\ B\ D$ and $G\ D\ C$ are both right angles, therefore $B\ D\ C$ is the complement of both $G\ D\ B$ and $B\ C\ D$, hence the angle $C =$ angle $G\ D\ B =$ dip.

Let r be the radius of the earth, h the height of the observer's eye above its surface, both in feet,—

$$\text{then cos. } C = \frac{B\,C}{C\,D} = \frac{r}{r+h};$$

$$\therefore 1 - 2\sin.^2\frac{C}{2} = \frac{r}{r+h}$$

$$2\sin.^2\frac{C}{2} = 1 - \frac{r}{r+h} = \frac{h}{r+h}.$$

Now h is so small compared with r, that it may be neglected in the denominator.

$$\text{Then } 2\sin.^2\frac{C}{2} = \frac{h}{r},$$

$$\sin.\frac{C}{2} = \sqrt{\frac{h}{2r}}.$$

$\frac{C}{2}$ being such a small angle, $\frac{C}{2} \times \sin. 1'$ may be written for $\sin. \frac{C}{2}$;

$$\therefore \frac{C}{2} \times \sin. 1' = \sqrt{\frac{h}{2r}},$$

$$\text{and } C \text{ in minutes} = 2\sqrt{\frac{h}{2r}} \times \frac{1}{\sin. 1'};$$

$$\text{or dip in minutes} = \sqrt{h} \times \frac{1}{\sin. 1'} \times \sqrt{\frac{2}{r}} \quad . \quad . \quad . \quad \text{I.}$$

Sir George Airy, the late Astronomer-Royal, gives:[1]—

Equatorial radius = 20923713 feet.
Polar radius = 20853810 „

If we take the radius to be a mean of these

$$r = 20888761{\cdot}5 \text{ feet}$$

$$\text{and } \frac{1}{\sin. 1'} = \frac{1}{{\cdot}000290888208} = 3437{\cdot}75,$$

we get from I—

$$\text{Dip in minutes} = \sqrt{h} \times 3437{\cdot}75 \times \sqrt{\frac{2}{20888761{\cdot}5}}$$

$$= \sqrt{h} \times \frac{3437{\cdot}75}{3231{\cdot}78}$$

$$= 1{\cdot}064\sqrt{h} \text{ very nearly.}$$

Dr. Nevil Maskelyne[2] says $\frac{1}{10}$ of the whole dip should be subtracted on account of refraction. Other authors give amounts varying from $\frac{1}{10}$ to $\frac{1}{15}$ of itself. If we use $\frac{2}{25}$ the result will perhaps approximate very closely to what it should be. This gives:—

$$\text{Dip in minutes} = \sqrt{h}\left(1{\cdot}064 - \frac{2}{25} \times 1{\cdot}064\right)$$

$$= {\cdot}9789\sqrt{h} \text{ in feet.}$$

From the above it is evident *the dip in minutes very nearly equals the square root of the height of the eye in feet.*

[1] Figure of the earth, "Encyclopædia Metropolitana."

[2] Astronomer-Royal from 1765 to 1811, and the one with whom the "Nautical Almanac" originated.

The student who prefers it may begin at

$$\text{Cos. } C = \frac{r}{r+h} = \frac{1}{1+\frac{h}{r}}$$

$$\therefore (1 - \sin.^2 C)^{\frac{1}{2}} = \left(1 + \frac{h}{r}\right)^{-1}$$

Expanding both sides by the binomial theorem

$$1 - \tfrac{1}{2}\sin.^2 C - \tfrac{1}{8}\sin.^4 C + \&c. = 1 - \frac{h}{r} + \frac{h^2}{r^2}, \&c.,$$

$\sin C$ and $\frac{h}{r}$ are such small fractions, we can without material error omit all terms after the second, then

$$\tfrac{1}{2}\sin.^2 C = \frac{h}{r}$$

and this reduces as before to

$$\text{dip in minutes} = \cdot 9789\sqrt{h} \text{ in feet.}$$

Ex. 82. Calculate the dip for a height of 18 feet.

$$\begin{aligned} \text{Dip in minutes} &= \cdot 9789\sqrt{18}. \\ \therefore \log. \text{ dip in minutes} &= \log. \cdot 9789 + \tfrac{1}{2}\log. 18. \\ &= \bar{1}\cdot 990738 + \cdot 627636. \\ &= 0\cdot 618374. \\ &= \log. 4\cdot 153. \\ \text{Hence dip for 18 feet} &= 4'\ 9\cdot 18''. \end{aligned}$$

Other heights should be selected, and after calculating the dips the answers should be compared with those from a good set of tables. A small variation will be found in tables by different authors, owing to the difference in the amount allowed for refraction.

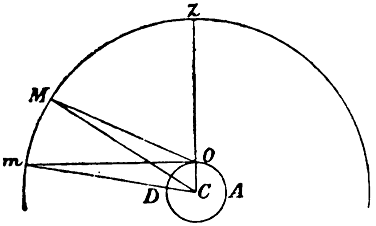
3. Correction for Semidiameter.—The sun, moon, and planets appear as circular discs, and as there are no points to mark their centres, it is almost impossible to measure, at sight, the altitudes of these bodies. The difficulty is obviated by observing either the upper or lower limbs of the objects, and applying the correction for semidiameter, which is given for the sun in the "Nautical Almanac," page ii. of each month, for every day at noon; for the moon at page iii., for every day at noon and midnight; and for each planet under its own heading. The semidiameter, there given, *is the angle at the centre of the earth subtended by the semidiameter of the object;* and because the observer, being on the surface, is nearer the object in every

position, except when it is in the celestial horizon, this angle requires a correction before it can be applied to the altitude.

AUGMENTATION is the term given to the correction of the semidiameter for the nearer position of the observer as the altitude of the object increases.

The sun's mean distance is about 93 millions of miles, and the earth's mean radius about 3956 miles, therefore his distance is about 23,500 radii of the earth; hence, if the sun be in the zenith of the observer, his semidiameter should be augmented by $\frac{1}{23500}$ of its tabulated amount, a quantity quite inappreciable. The moon's mean distance being about 238,000 miles, her distance when in the horizon will be only about 60 of the earth's radii; but when in the zenith she must be one radius nearer, or 59 radii distant, and hence her semidiameter must be augmented in the ratio $\frac{60}{59}$ of its value in the horizon. In every other position from the horizon to the zenith her semidiameter must have a corresponding augmentation, which is found thus:—

Fig 15



Let $A\ O\ D$ be a section of the earth through its centre C, m the moon in the horizon of the observer situate at O, M the moon in a circle of altitude, Z the zenith, join $m\ C$, $m\ O$, $M\ C$, $M\ O$, and $Z\ O\ C$.

Now the semidiameter, when the moon is at m, is so near that given in the "Nautical Almanac" that we may take one for the other. Let s be this semidiameter, and a the augmentation sought when the moon is at M, then $s + a$ will be her semidiameter, and this will be greater than s in the ratio of $M\ C$ to $M\ O$.

$$\therefore \frac{M\ C}{M\ O} = \frac{s + a}{s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{I.}$$

Let $Z\ O\ M$ the apparent zenith distance $= z$, and $C\ M\ O$ the parallax $= p$.

Then $Z\ C\ M$ the true zenith distance $= Z\ O\ M - C\ M\ O = z - p$.

$$\text{Now} \frac{M\ C}{M\ O} = \frac{\sin.\ C\ O\ M}{\sin.\ O\ C\ M} = \frac{\sin.\ Z\ O\ M}{\sin.\ Z\ C\ M}$$

$$= \frac{\sin.\ z}{\sin.\ (z - p)} \quad . \quad . \quad . \quad . \quad \text{II.}$$

Equating I. and II.

$$\frac{s+a}{s} = \frac{\sin. z}{\sin.(z-p)} \quad \text{subtract 1 from each side}$$

$$\frac{a}{s} = \frac{\sin. z - \sin.(z-p)}{\sin.(z-p)} = \frac{2\cos.\frac{2z-p}{2}\sin.\frac{p}{2}}{\sin.(z-p)};$$

$$\therefore a = 2s\cos.\frac{2z-p}{2}\sin.\frac{p}{2}.\ \text{cosec.}(z-p)\ .\quad \text{III.}$$

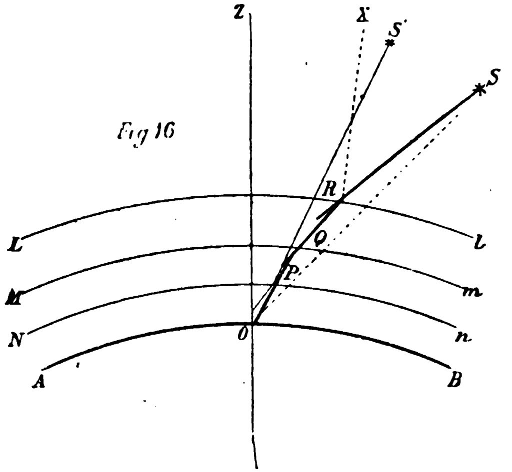
This is the formula used in calculating the augmentation of the moon's semidiameter, and registered in all good nautical tables. The value of p is found on page 59 from the formula.

Par. in alt. = hor. par. × cos. app. alt.

Hence to correct an altitude for the *sun's* semidiameter we must apply the value given in the "Nautical Almanac;" but the *moon's* semidiameter must first be augmented by the quantity a found above in equation III. If the *lower* limb be observed we must *add* the correct semidiameter, because it is evident the altitude of the sun's centre must be greater than that of the lower limb; but, if the *upper* limb be observed, the correct semidiameter must be *subtracted* from the altitude already corrected for index error and dip.

4. CORRECTION FOR REFRACTION.—Refraction is the bending of rays of light from their direct courses, owing to their passing through media of different densities. The earth is surrounded by an atmosphere of variable density; hence rays of light proceeding from the heavenly bodies are bent or refracted from their direct courses before entering the eye of the observer.

Let *A O B* be a portion of a section of the earth through its centre; the concentric circles *L R l*, *M Q m*, &c., strata of air supposed to be of uniform density throughout each stratum, but increasing in density as they approach the earth. Then a ray from *S*, traversing space before entering the highest layer, proceeds in a straight line *S R*; at *R* it is bent towards the vertical *X R*, and instead of proceeding on in the same straight line *S R*, it takes another direction as *R Q*, nearer to the vertical *X R*. At the surface of the next layer it is again deflected, and takes a direction nearer again to the vertical at *Q*, as *Q P*, and so on at the surface of each stratum, until it reaches the observer's eye at *O*, and then it appears to come to him in the direction *S′ O*, or the direction in which the last part of the ray



comes to the eye. If we take each stratum indefinitely thin, and the number of them indefinitely great; then, instead of angles being formed in the course of the ray through the atmosphere, it will form a continuous curve, and the object seen will appear in the direction of the tangent to the curve at the point where the ray enters the eye. The angular difference between the tangential direction $O\ S'$, and that in which the ray would have come in a straight line $O\ S$ from the object to the eye, that is, the angle $S\ O\ S'$ is the amount of refraction.

In physical optics it is shown that when a ray of light enters a dense medium from one not so dense, it is bent towards the normal of the surface separating the media; hence, except when in the zenith, an object appears of greater altitude than it really is; and therefore the correction for refraction must always be subtracted from the apparent altitude of the object.

LAWS OF REFRACTION.—It may be well, here, to recall to the student's memory the two laws of refraction, which were first discovered by Willibrord Snell (1591—1626), and are as follows:—

(*a*) *Coincidence of the planes of refraction.*—The angles of refraction and incidence are in the same plane, which plane contains the normal to the refracting surface.

(*b*) *The index of refraction.*—The sine of the angle of inci-

dence has to the sine of the angle of refraction a constant ratio for the same medium.

FORMULÆ FOR REFRACTION—BESSEL'S AND BRADLEY'S.

Let $r =$ the mean refraction in seconds.
$c =$ the constant ratio or index of refraction.
$z =$ the angle of refraction.
Then $z + r =$ the angle of incidence.

We shall have by the second law above:—

$$\text{Sin.}\ (z + r) = c\ \text{sin.}\ z.$$

$$\text{Sin.}\ z.\ \cos.\ r + \cos.\ z.\ \text{sin.}\ r = c.\ \text{sin.}\ z.$$

But as the refraction is always small, being when at a maximum only about 34′, we may take $\cos.\ r =$ unity

and $\text{sin.}\ r = r.\ \text{sin.}\ 1''$.

$$\therefore \text{sin.}\ z + r.\ \text{sin.}\ 1''.\ \cos.\ z = c.\ \text{sin.}\ z$$

and
$$r = \frac{(c-1)\ \text{sin.}\ z}{\text{sin}\ 1''.\ \cos.\ z}$$

$$= \frac{c-1}{\text{sin.}\ 1''}\ \tan.\ z \quad . \quad . \quad . \quad . \quad \text{I.}$$

In the case of astronomical observations z is approximately the zenith distance of the object. We see, therefore, the mean amount of refraction (in altitudes of more than 12° or 13°) varies very nearly as the tangent of the zenith distance, and is

$$= \text{constant} \times \tan.\ \text{zen. dist. nearly.}$$

That the refraction increases with the zenith distance might have been anticipated if we consider that when the altitude of a heavenly body is small, therefore its zenith distance great, the rays of light from that object have to pass through a thicker layer of atmosphere of variable density than when the rays come from an object which is vertical; and thus they will become bent further from their paths.

From the formula which Bradley deduced for the amount of refraction he found a more exact result than that found in I. is given by

$$r = \frac{c-1}{\text{sin.}\ 1''}\ \tan.\ (z - 3r),$$

and from repeated observations on circumpolar stars Bessel gave the value 57·538″ to the constant $\frac{c-1}{\text{sin.}\ 1''}$; and the formula for the mean refraction finally assumes the form

$$r = 57{\cdot}538''.\ \tan.\ (z - 3\ r)\ .\ .\ .\ .\ .\ .\quad \text{II.}$$

and is the one from which tables for mean refraction are calculated.

Ex. 83. Using the above formula, find the mean refraction for an altitude of 37° 30′.

Here $z = 52°\ 30'$. $\therefore$ approximately $r = 57{\cdot}538''.\ \tan.\ 52°\ 30'$.

$$\begin{aligned}
\text{Log. } r &= \text{log. } 57{\cdot}538'' + \text{log. tan. } 52°\ 30'. \\
&= 1{\cdot}759954 + 10{\cdot}115020 - 10. \\
&= 1{\cdot}874974. \\
&= \text{log. } 74{\cdot}985'' \text{ a first approximation.} \\
\therefore\ 3\ r &= 224{\cdot}955'' = 3'\ 45'' \text{ very nearly.}
\end{aligned}$$

Again:—

$$\begin{aligned}
r &= 57{\cdot}538''.\ \tan.\ (52°\ 30' - 3'\ 45''). \\
&= 57{\cdot}538''.\ \tan.\ 52°\ 26'\ 15''. \\
\therefore\ \text{log. } r &= \text{log. } 57{\cdot}538'' + \text{log. tan. } 52°\ 26'\ 15''. \\
&= 1{\cdot}759954 + 10{\cdot}114039 - 10. \\
&= 1{\cdot}873993. \\
&= \text{log. } 74{\cdot}82'' \text{ a second approximation.}
\end{aligned}$$

The two approximations agreeing so nearly it is unnecessary to carry it any further; but by so doing any degree of precision may be attained.

$$\therefore\ r = 1'\ 14{\cdot}8''$$

which agrees very nearly with the table in the "Encyclopædia Metropolitana," page 544. As an exercise in the use of the formula II. the student should select other examples for himself, and refer to tables for the verification of his work.

MODIFICATION OF FORMULÆ.—It has been found that refraction varies with the density of the medium through which a ray of light passes; and because the atmosphere is constantly varying in density with its pressure, temperature, and (probably) with its hygrometric[3] condition, the *mean refraction*, calculated from the above formula, must be modified for the density of the air. Marriott's or Boyle's law states—"The pressure of a given quantity of air varies inversely as the space it occupies," and also "the density varies directly as the pressure." The pressure is measured by the barometer. Bradley's and Bessel's combined formula (II.) is formed for the density corresponding to 29·6 inches of mercury in the barometer, and a temperature

[3] La Place, Gay Lussac, and Biot say "refraction is not influenced by the relative moisture of the atmosphere."—*Woodhouse's "Astronomy,"* p. 235.

of 50° Fahr. From the above law, if r be the mean refraction, h the height of the barometer, we shall have:

$$\text{Refraction corrected for pressure} = \frac{h}{29{\cdot}6} \cdot r \quad . \quad . \quad \text{III.}$$

Dalton and Gay Lussac found that gases expand ·3665 of their volume for an increase of temperature from freezing to boiling point of water, or for an increase of 180° Fahr. Hence a quantity of air at a temperature of 50° Fahr. will under the same pressure on being raised to $t°$ Fahr. have expanded $\frac{{\cdot}3665}{180}(t° - 50°) = {\cdot}002036\ (t° - 50°)$; and its volume must then be $1 + {\cdot}002036\ (t° - 50°)$ of its former volume. Now its density, by Boyle's law, will be inversely as its volume, and hence equation III. must be multiplied by $\frac{1}{1 + {\cdot}002036\,(t° - 50°)}$ for the true refraction.

$$\therefore \text{True refraction} = \frac{1}{1 + {\cdot}002036\ (t - 50)°} \times \frac{h}{29{\cdot}6} \times 57{\cdot}538''\,\tan.(z - 3r)$$

$$= \frac{491{\cdot}1}{441{\cdot}1 + t°} \times \frac{h}{29{\cdot}6} \times 57{\cdot}538''.\ \tan.(z - 3r)\ .\ \text{IV.}$$

which agrees very nearly with Bradley's formula, which is:—

$$\text{True refraction} = \frac{400}{350 + t°} \times \frac{h}{29{\cdot}6} \times 57{\cdot}538''.\ \tan.(z - 3r);$$

and Delambre's:—

$$\text{True refraction} = \frac{500}{450 + t°} \times \frac{h}{29{\cdot}6} \times \left\{ \begin{array}{l} 60''\tan.z - {\cdot}14207. \\ \tan.^3 z - {\cdot}0045053. \\ \tan.^5 z,\ \&c. \end{array} \right\}$$

The corrections of the mean refraction for the heights of the barometer and thermometer, placed in most nautical tables, are derived from equations III. and IV.

For very exact observations, but never required in Nautical Astronomy, a correction for expansion of the mercury is used, and is $\frac{1}{5500}$ for 1° C. or $\frac{1}{9900}$ for 1° Fahr. (Besant's "Hydrostatics").

UNCERTAINTY OF THESE LAWS.—Woodhouse says in his "Astronomy," page 223: "The uncertainty respecting the correction of refraction for difference of temperature is rather an embarrassing circumstance when minute inequalities are to be detected." And again, page 242: "Hitherto there has been invented no

formula that restricts the irregularity of refraction that begins to take place about 80° of zenith distance; and those bordering on 90° are freed from all restraint. They disagree among themselves, and are not the same when other circumstances, viz. the altitude and heights of the barometer and thermometer, are the same. It is certain, then, that the theory of refraction is imperfect, not solely because it does not restrict all its cases within the same law, but because it has no test of, or means of measuring, certain circumstances, on which, at great zenith distances, the refraction must depend. This is a perplexity from which mathematical skill alone can never extricate us."

Effects of Refraction on the Diameter of Objects.—When objects with a large disc, as the sun and moon, are near the horizon it is evident from what has already been said that refraction will elevate the lower limb more than the upper, and hence the vertical diameter will appear contracted and the disc will assume an oval form. This may be seen from the following example.

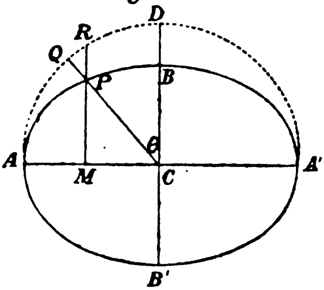
Ex. 84. Suppose the moon's diameter from the "Nautical Almanac" to be 33′ 30″, and the altitude of the moon's lower limb to be 3°, find the contraction of her vertical diameter.

	Lower limb.	Upper limb.
Altitude	3° 0′ 0″	3° 33′ 30″
Mean Refraction	14 36	12 57
	3 14 36	3 46 27

The difference is 31′ 49″, which is the apparent vertical diameter. Hence the contraction is the difference between this and the true diameter, 33′ 30″, viz. 1′ 41″. In finding the index error of a sextant we should therefore never use the sun when his altitude is low, unless his horizontal diameter be observed for the purpose. The horizontal diameter also undergoes a slight change in magnitude, as may be seen at once; for the extremities of that diameter are each elevated in its own vertical circle, and as these circles all meet in the zenith, it is evident the horizontal diameters must also slightly contract with the zenith distance. All the other diameters will also contract in the ratio of the maximum contraction multiplied by the cosine squared of the angle made with the vertical diameter.

This may be shown as follows:—

Fig 17.



Let $A\ C\ A'$ be the horizontal diameter of the oval disc.

$B\ C\ B'$ the vertical one.

$C\ P$ the semidiameter, making an angle θ with the vertical diameter.

$P\ M$ the ordinate at P.

Produce $C\ B$, $C\ P$, and $M\ P$ to meet the auxiliary circle in $D\ Q$ and R: then the contractions of the ordinates are approximately proportional to their magnitudes—

$$\text{i. e. } P R : B D :: M P : C B.$$
$$:: M P : C P \text{ approximately.}$$
$$\therefore P R = B D \cos. \theta.$$

Again, the small triangle $P\ Q\ R$ is right-angled at Q.

$$\therefore P Q = P R \cos. \theta.$$
$$\text{Hence } P Q = B D \cos.^2 \theta.;$$

that is, the contraction of any semidiameter is equal to that of the vertical semidiameter multiplied by the square of the cosine of the included angle.[4]

5. CORRECTION FOR PARALLAX.—In all the preceding investigations no notice has been taken of the position of the observer, whether at the surface or the centre of the earth; because the distances of the heavenly bodies are so great compared with the earth's radius, that (in the corrections already explained) no material error is introduced by taking into consideration where he is situated. But there are several bodies, as the sun, moon, planets, and comets, whose distances, compared with the radius of the earth, cannot be neglected, because their positions, when projected back among the fixed stars, will appear different when viewed from different parts of the earth's surface. To secure uniformity of position, and for comparison with other observations when made, they are reduced to what they would appear at some definite point; and because the centre of the earth is the most convenient for such reduction, and also the apparent motions of the heavenly bodies are of a much more simple character when viewed from there than from the surface,

[4] Godfray's "Astronomy."

that point is selected as the most advantageous for reference.

The APPARENT PLACE OF A HEAVENLY BODY is its position on the celestial concave as seen from where the observer is situated.

The TRUE PLACE OF A HEAVENLY BODY is its position on the celestial concave as seen from the centre of the earth when unaffected by refraction.

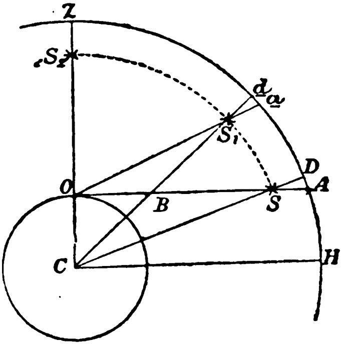
ANNUAL OR HELIOCENTRIC PARALLAX is the difference of place of a heavenly body on the celestial concave, seen respectively from two opposite points in a diameter of the earth's orbit, or it is the angle at the body subtended by a diameter of the earth's orbit.

DIURNAL or GEOCENTRIC PARALLAX is the difference of place of a heavenly body on the celestial concave seen respectively from the surface and from the centre of the earth; or it is the angle at the object's centre subtended by the radius drawn from the observer to the earth's centre.

Nautical Astronomy deals only with the latter, and takes no cognizance of the former.

Let C be the centre of the earth, O the place of an observer on its surface, $O\ A$ the sensible horizon, $C\ H$ the rational horizon, $S\ S_1\ S_2$ the sun, moon, planet, or comet in three positions; $H\ A\ D\ a\ d\ Z$ a vertical circle on the celestial concave (in the plane that passes through C, O, and S) on which the body is seen projected. Then at S in the horizon, the radius of the earth $O\ C$ subtends the angle $O\ S\ C$, therefore $O\ S\ C$ is the horizontal parallax; similarly at S_1 the angle $O\ S_1\ C$ is the parallax in altitude, but at S_2, where the object is in the zenith, the radius $O\ C$ appears at a point, and because no angle is there subtended by the radius $O\ C$, therefore in such a case there is no parallax. A is the apparent place of S, and D its true place; similarly a is the apparent place of S_1, and d its true place.

Fig 18.



Again, Eu. I. 21, it is manifest the farther an object is from

the earth in the same direction, the less is the angle which the radius of the earth subtends; thus the moon, being the nearest of all celestial bodies, will have the greatest parallax; and the fixed stars, being at such an immense distance, that the whole earth is not seen at all, will have no parallax.

If S_1 be the position of the object seen by the observer at O,

$S_1\,O\,A$ is the apparent altitude of S_1

$S_1\,C\,H$ is the true altitude of S_1

Now $S_1\,C\,H = S_1\,B\,A = S_1\,O\,B + O\,S_1\,B$,

or true alt. = app. alt. + par. in alt.

Hence, to get the true altitude we must add the parallax in altitude to the apparent altitude, and when the object is in the zenith the true and apparent altitudes coincide; and it is manifest there is then no parallax. The parallax must be a maximum when a straight line, drawn from the object perpendicular to the radius of the earth, bisects it.

VARIATION OF HORIZONTAL PARALLAX.—Because the earth is an oblate spheroid, the equatorial radius being about thirteen miles longer than the polar, the horizontal parallax, as seen from the equator, will be greater than the horizontal parallax of the same body when seen at the same distance from the terrestrial poles, and will diminish as the latitude increases. The parallax of the moon given in the "Nautical Almanac" is *the equatorial horizontal parallax;* hence, for observations of the moon, two corrections are necessary before we get the true parallax in altitude, viz.—

(1) For the altitude of the object;
(2) For the latitude of the observer;

while for the sun the former only of these corrections is necessary, because of his greater distance.

The first correction is easily computed from the horizontal parallax thus—

In triangle $S_1\,O\,C$, $\dfrac{O\,C}{C\,S_1} = \dfrac{\sin.\ O\,S_1\,C}{\sin.\ C\,O\,S_1}$;

but $C\,S = C\,S_1$ and $C\,O\,S_1 = \pi - Z\,O\,S_1$;

$$\therefore \frac{O\,C}{C\,S} = \frac{\sin.\ O\,S_1\,C}{\sin.\ Z\,O\,S_1}$$

$$\text{Hence } \sin.\ O\,S\,C = \frac{\sin.\ O\,S_1\,C}{\cos.\ S_1\,O\,A},$$

$$\text{or } \sin.\ \text{hor. par.} = \frac{\sin.\ \text{par. in alt.}}{\cos.\ \text{app. alt.}}.$$

Now as the horizontal parallax [5] and therefore the parallax in altitude are both very small, we may write

$$Hor.\ par.\ in\ sec. \times sin.\ 1'' \text{ for } sin.\ hor.\ par.$$

and similarly for par. in alt.

$$\therefore hor.\ par.\ in\ sec. \times sin.\ 1'' = \frac{par.\ in\ alt.\ in\ sec. \times sin.\ 1''}{cos.\ app.\ alt.}$$

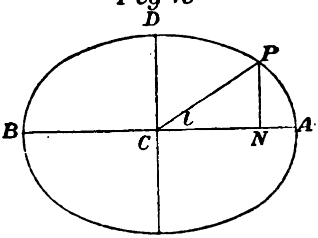
i.e. $par.\ in\ alt.\ in\ sec. = hor.\ par.\ in\ sec. \times cos.\ app.\ alt.$

The second correction alluded to is called—

Reduction of the Moon's Horizontal Parallax.—At the same distance of the body from the centre of the earth, the parallax varies as the radius vector of the spheroid. A table, therefore, that gives the several values of the radii in a spheroid of a given oblateness, enables us to correct the equatorial parallax. Such a table, calculated for a compression of $\frac{1}{300}$ and $\frac{1}{330}$, is given by Vince in his "Astronomy," vol. iii., table xiv.

Let $A\ D\ B\ E$ be a section of the earth through its poles D and E; P any point on its surface whose latitude $P\ C\ A = l$.

Fig 19



The equation to the generating ellipse is $b^2\ x^2 + a^2\ y^2 = a^2\ b^2$ where $A\ C = a, D\ C = b, C\ N = x = P\ C.\ cos.\ l$ and $P\ N = y = P\ C.\ sin.\ l.$

The equation may then be written—

$$b^2.CP^2.cos.^2l + a^2.CP^2.sin.^2l = a^2b^2$$

$$CP^2 = \frac{a^2\ b^2}{b^2 \cos.^2 l + a^2 \sin.^2 l};$$

dividing each side by a^2 and taking the square root of each side we get

$$\frac{CP}{a} = \frac{b}{\sqrt{b^2 + a^2 e^2 \sin.^2 l}},$$

$$\text{because } e^2 = \frac{a^2 - b^2}{a^2}$$

$$= \left(1 + \frac{a^2\ e^2}{b^2} \sin.^2 l\right)^{-\frac{1}{2}};$$

expanding by the binomial theorem, and considering that e is so small (about $\frac{3}{37}$), and that $sin.\ l$ is a proper fraction, we may

[5] From observations made at Paris, Leland computed the maximum hor. par. of the moon to be 61′ 29″.—*Woodhouse's "Astronomy,"* p. 652.

omit all terms containing the product of powers of e and $sin.\ l$ above the second—

$$\text{then } \frac{CP}{a} = 1 - \frac{a^2 e^2}{2b^2} \sin.^2 l \text{ very nearly}$$

$$= 1 - \frac{a^2}{2b^2} \times \frac{a^2 - b^2}{a^2} \sin.^2 l$$

$$= 1 - \frac{a-b}{b} \sin.^2 l \text{ very nearly} \quad . \quad . \quad \text{I.}$$

because $a + b$ very nearly equals $2b$.

Let H be the moon's equatorial horizontal parallax, and h the horizontal parallax at P in latitude l;

$$\text{then } \frac{CP}{a} = \frac{hor.\ par.\ for\ lat.\ P}{equa.\ hor.\ par.} = \frac{h}{H} \quad . \quad . \quad . \quad \text{II.}$$

Equating I. and II.

$$h = H - H\frac{a-b}{b} \sin.^2 l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{III.}$$

that is the equatorial horizontal parallax must be diminished by a quantity $\frac{a-b}{a} \sin.^2 l$ of itself, in order to find the horizontal parallax for a place P in latitude l. The factor $\frac{a-b}{b}$ depends upon the form of the spheroid, which in the case of the earth is about $\frac{1}{300}$.

Equation III. finally assumes the form $h = H - H\frac{\sin.^2 l}{300}$; the latter term is called the reduction of the moon's equatorial horizontal parallax.

Ex. 85. If the horizontal parallax of the moon taken from the "Nautical Almanac" be 57′, what will be the horizontal parallax for the latitude of the Navigation School, Plymouth?

$$\text{Reduction} = \frac{H}{300} \sin.^2 l.$$

Latitude 50° 22′ 25″	sin. = 9·886614	
	2	
sin. $^2 l$	9·773228	
$\frac{H}{300} = \frac{57'}{300} = \frac{11{\cdot}4'}{60} = 11{\cdot}4''$ log.	1·056905	
Reduction	0·830133 =	0′ 6·763″
Equatorial horizontal parallax		= 57 0
Hence reduced horizontal parallax		= 56 53·237

The equatorial horizontal parallax $O\ S\ C$ (Fig. 18) for the moon is given in page 3 of the "Nautical Almanac" for noon and midnight for every day in the year; and as the sun is at a comparatively great distance (23500 radii of the earth) his horizontal parallax never varies sensibly from 8·8″; but that for the moon changes very much from the fact that she oscillates, as it were, between a distance varying from 56 to 63 radii, and hence her parallax will vary as much as one-eighth part of itself.

Because the mean refraction and the parallax both depend for their values on the altitude of the sun, and the refraction is always greater than the parallax, a table is inserted in some collections, entitled, "*Correction of the Sun's Apparent Altitude.*" It is formed by subtracting the sun's parallax in altitude from his mean refraction for that altitude, and registering the remainder as the correction. This saves labour, as the one operation then suffices for the two. In the case of the moon no such table can be formed, because the reduction of her horizontal parallax depends for its value on the latitude of the observer.

Parallax is especially interesting to the astronomer, because it presents the *only* method of determining the distances and dimensions of celestial objects. Those whose distances have an appreciable ratio to the earth's radius, as the moon, the sun, and the planets, can have their distances measured by means of *diurnal parallax;* but those whose distances are so great as to render this method inapplicable have theirs determined (when possible) by *annual parallax.* The distances of these with diurnal parallax are found as follows:—

Ex. 86. Find the moon's distance from the earth when her equatorial horizontal parallax is 57′ 33″.

Royal Naval College, 1865.

The earth's equatorial radius is 3963·25 miles; and as the angle subtended by this radius is so small we can at once apply the formula from circular measure—

$$a = \theta\, r.$$

where $a = 3963{\cdot}25$ miles, $\theta =$ the circular measure of 57′ 33″ and $r =$ the distance of the moon. Then—

$$\begin{aligned} r = \frac{a}{\theta} &= \frac{3963{\cdot}25}{57'\ 33''} \times \frac{180^\circ}{\pi} \text{ miles} \\ &= \frac{3963{\cdot}25 \times 180 \times 60 \times 60}{3453 \times 3{\cdot}14159265} \\ &= 236745 \text{ miles.} \end{aligned}$$

This question may also be solved without assuming the formula $a = r\,\theta$ thus:—

The angle subtended by the radius of the earth is so small that we may consider that radius as a small arc of the circle described from the moon's centre, with her distance as radius. Then—

$$\text{Circumference of this circle} = \frac{360^\circ}{57'\ 33''} \times 3963{\cdot}25 \text{ miles};$$

but the circumference also $= 2\pi r$;

$$\therefore r = \frac{360 \times 60 \times 60 \times 3963{\cdot}25}{3453 \times 2 \times 3{\cdot}14159265} \text{ miles}$$
$$= 236745 \text{ miles as before.}$$

Having found the distance by means of the parallax, the dimensions of the body are determined by its semidiameter thus:—

Ex. 87. Find the real diameter of the moon when her semidiameter from the "Nautical Almanac" is $15'\ 42{\cdot}5''$, taking her distance from the earth as found in the last example.

$$a = \theta\, r$$
$$= \frac{15'\ 42{\cdot}5'' \times \pi}{180} \times 236745 \text{ miles}$$
$$= \frac{942{\cdot}5 \times 3{\cdot}14159265 \ldots \times 236745}{180 \times 60 \times 60}$$
$$= 1081{\cdot}8 \text{ miles};$$
$$\therefore \text{Diameter} = 2163{\cdot}6 \text{ miles.}$$

This question, like the last, may be solved without using the formula $a = \theta r$.

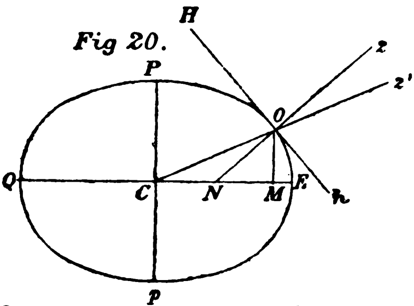
If we bear in mind that the distances of a movable object vary inversely as the linear measurement of one of its dimensions; then if we carefully measure the diameter of the moon throughout a lunation, or one of her revolutions around the earth, and lay down to scale the distances obtained by taking the inverse of the measured diameters, we shall have a rough proof that the orbit of the moon is elliptical. Similar remarks apply to the sun's orbit.

REDUCED LATITUDE AND ANGLE OF THE VERTICAL.—As the earth is not a perfect sphere but an oblate spheroid, the normal, i.e. the perpendicular to the sensible horizon at any place, except at the equator and poles, will not pass through the centre of the earth.

Let $P\,Q\,p\,E$ be a meridianal section of the earth, $P\,p$ the poles, $E\,Q$ the points where the plane of the equator cuts the sec-

tion, C the centre of the earth, O the place of the observer, $H\,O\,h$ his sensible horizon, which will be a tangent at O. Through O draw $Z\,O\,N$ perpendicular to $H\,O\,h$; $O\,N\,E$ measures the *true* or *astronomical* or *latitude by observation* of O; join $C\,O$ and produce $C\,O$ to Z^1, then $O\,C\,E$ measures the *reduced* or *geocentric latitude* of O; Z is the *true zenith*, Z^1 the *reduced zenith*, and the angle $Z\,O\,Z^1$ is the *angle of the vertical* or *reduction for latitude.*

Fig 20.



Let $a =$ equatorial radius $C\,E$ . . $b =$ polar radius $C\,P$;
$l =$ true latitude $O\,N\,E$. . . $l' =$ reduced latitude $O\,C\,E$

$C\,M = x$ . . . then $M\,N$ the subnormal is $\frac{b^2}{a^2}x$.

$$\text{Now tan. } l = \frac{O\,M}{M\,N} \text{ and tan. } l' = \frac{O\,M}{M\,C}.$$

$$\frac{\text{tan. } l'}{\text{tan. } l} = \frac{M\,N}{M\,C} = \frac{\frac{b^2}{a^2}\cdot x}{x} = \frac{b^2}{a^2}.$$

$$\text{Hence tan. } l' = \frac{b^2}{a^2}\text{ tan. } l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{I.}$$

$$\text{Again, tan. } (l - l') = \frac{\text{tan. } l - \text{tan. } l'}{1 + \text{tan. } l \text{ tan. } l'}$$

$$= \frac{\text{tan. } l\left(1 - \frac{b^2}{a^2}\right)}{1 + \frac{b^2}{a^2}\text{ tan. }^2 l}$$

$$= \frac{e^2 \text{ tan. } l}{1 + \frac{b^2}{a^2}\text{ tan. }^2 l} \quad . \quad . \quad . \quad . \quad \text{II.}$$

Now $\frac{b^2}{a^2}$ is so nearly equal to unity, that no material error will be introduced into the calculation if we let $\frac{b^2}{a^2}$ in the denominator $= 1$. Then

$$\tan.(l - l') = \frac{e^2 \tan. l}{1 + \tan.^2 l} \text{ very nearly}$$

$$= \frac{e^2}{2} \frac{2\dfrac{\sin. l}{\cos. l}}{1 + \dfrac{\sin.^2 l}{\cos.^2 l}}$$

$$= \frac{e^2}{2} \sin. 2l \quad . \quad . \quad . \quad . \quad . \quad \text{III.}$$

Now $\quad (l - l') = ONE - OCE = NOC$
$\qquad\qquad\qquad = \text{angle of the vertical.}$

and because $l - l'$ only amounts (when at its maximum) to a few minutes, the tangent and the arc are very nearly equal.

From III. $\therefore$ angle of the vertical $= \frac{e^2}{2} \sin. 2l$ very nearly.

Again, $\frac{e^2}{2}$ is a constant for the earth, therefore *the angle of the vertical varies as sine of twice the latitude*, hence it will be a maximum when $2l$ is so, i.e. when the latitude is 45°.

Returning to Formula II., by giving to a and b the values found by the late Astronomer-Royal, viz. $a = 3962{\cdot}824$, and $b = 3949{\cdot}585$ miles, we can find the maximum value of the angle of the vertical thus :—

$$\tan.(l - l') \text{ or tan. angle of the vertical} = \frac{e^2 \tan. l}{1 + \frac{b^2}{a^2} \tan.^2 l};$$

but the angle has just been shown to be a maximum when $l = 45°$,

then $\tan. l = 1$; and because $e^2 = \frac{a^2 - b^2}{a^2}$ we get

$$\text{tan. angle of the vertical} = \frac{a^2 - b^2}{a^2 + b^2};$$

and giving a and b their numerical values, we find the maximum value of the angle of the vertical $= 11'\ 30{\cdot}24''$.

Ex. 88. Find the reduced latitude and the angle of the vertical for the Plymouth Navigation School, latitude 50° 22′ 25″ N.

$$\tan. l' = \frac{b^2}{a^2} \tan. l;$$

$$\therefore \log. \tan. l' = 2(\log. b - \log. a) + \log. \tan. l,$$

or log. tan. reduced lat. $= 2$ (log. $3949{\cdot}585$ − log. $3962{\cdot}824$) + log. tan. $50° \; 22' \; 25''$

$$= 2\,(3{\cdot}596551 - 3{\cdot}598005) + 10{\cdot}081944$$
$$= \bar{1}{\cdot}997092 + 10{\cdot}081944$$
$$= 10{\cdot}079036$$
$$= \text{log. tan. } 50° \; 11' \; 6{\cdot}1''.$$

∴ Angle of the vertical of the school

$$= 50° \; 22' \; 25'' - 50° \; 11' \; 6{\cdot}1''$$
$$= 11' \; 18{\cdot}9''.$$

This gives very nearly the same result for the angle of the vertical as if we had used equation III. giving e the value $\frac{3}{37}$; but the present result is more correct, because in that case $\frac{b^2}{a^2}$ was taken equal to unity.

The log. $\frac{b^2}{a_2}$ may be registered (it being equal to $\bar{1}{\cdot}997092$, as shown above) which would greatly facilitate the calculations; the formula would then be

log. tan. reduced lat. $= \bar{1}{\cdot}997092$ + log. tan. lat. by observation.

Angle of the Vertical at any Azimuth.—When the moon is on the meridian of the observer at her upper transit, the zenith distance should be diminished or the apparent altitude increased by the whole amount as found above, before the parallax in altitude is computed from the formula,

Par. in alt. in sec. = hor. par. in sec. × cos. app. alt.

When the moon is on the meridian below the pole, the correction should be applied in a contrary manner.

The last formula applies only to altitudes on the meridian, because $P \, E \, p \, Q$ (Fig. 20) was supposed to be a meridional section of the earth: we will now investigate a formula for finding the angle of the vertical at all times.

Let N E S W be the plane of the horizon, $N \, P \, S$ the meridian, P the elevated pole, Z the true zenith, Z' the reduced zenith (agreeing with Z and Z' in the last figure), G the position of the celestial object; join $Z \, G$, $Z' \, G$ by arcs of great circles, and from Z' draw $Z' \, K$ perpendicular to $G \, Z$, or $G \, Z$ produced.

Because $Z \, Z'$ is so small, being, as already proved, never greater than $11' \; 30{\cdot}24''$, we may take $Z \, K \, Z'$ as a plane triangle, and $Z \, K$ approximately the difference between $Z \, G$ and $Z' \, G$.

Then $Z \, Z'$ measures the angle of the vertical when the object

is on the meridian, and ZK is the correction required instead of that angle when the object has a given azimuth SZG.

Fig 21.

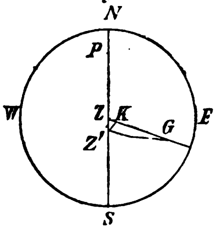
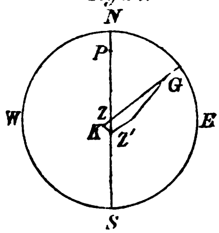


Fig 22.



In Fig. 21 $ZK = ZZ'$, cos. $Z'ZK = ZZ'$, cos. SZG.

In Fig. 22 $ZK = ZZ'$, cos. $Z'ZK = - ZZ'$, cos. SZG.

The latter is a positive quantity, because in Fig. 22, SZG is greater than 90°, hence its cosine is negative, and therefore

Correction = angle of vert. on meridian × cos. azimuth.

Hence, when the moon is not on the meridian, the azimuth should be observed or computed, and the correction from above be applied to the zenith distance.

In Fig. 22, where the azimuth is greater than 90°, it will be seen $Z'K$ falls on GZ produced, and therefore the correction ZK is additive to the observed zenith distance ZG to obtain the reduced zenith distance $Z'G$; but in Fig. 21, where the azimuth is less than 90°, the correction ZK is subtractive from the observed zenith distance; while, when the object is on the prime vertical, the correction is zero.

DIURNAL PARALLAX FOUND FROM OBSERVATIONS ON THE SAME MERIDIAN.—Let ABE be a meridional section of the

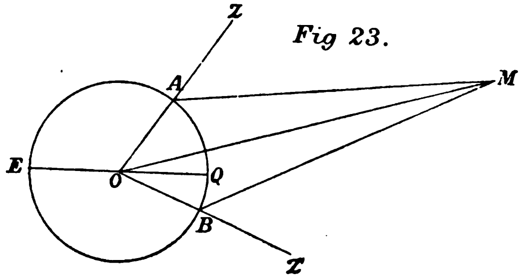


Fig 23.

earth, O its centre, E and Q points where the equator meets

the section. Let A and B be two observatories on the meridian whose latitudes are $A\,O\,Q = l$ and $B\,O\,Q = l'$, then Z is the zenith of A, and Z' of B, and let M be the position of the moon or other object whose parallax is sought, join $M\,A$, $M\,B$, and $M\,O$.

At A let the radius of the earth $O\,A = r$.
the reduced zenith distance $Z\,A\,M = z$,
and the parallax in alt. $A\,M\,O = p$.
At B let the radius of the earth $O\,B = r'$,
the reduced zenith distance $Z'\,O\,M = z'$,
and parallax in altitude $B\,M\,O = p'$,

$$\text{then } A\,M\,B \text{ or } p + p' = (Z\,A\,M - A\,O\,M) + (Z'\,B\,M - B\,O\,M)$$
$$= (Z\,A\,M + Z'\,B\,M) - (A\,O\,M + B\,O\,M)$$
$$= (z + z') - (l + l') \quad . \quad . \quad . \quad . \quad \text{I.},$$

a known quantity, because the latitudes of the places are known, and the zenith distances of M have been measured at the observatories A and B

Now if R be the distance $O M$ of the body from the centre of the earth, remembering that p and p' are small, we have from the figure

$$\frac{r}{R} = \frac{\sin. p}{\sin. z} \qquad \therefore\ p \sin. 1'' = \frac{r}{R} \sin. z,$$

$$\text{and } \frac{r'}{R} = \frac{\sin. p'}{\sin. z'} \qquad \therefore\ p' \sin. 1'' = \frac{r'}{R} \sin. z'.$$

$$\text{Hence—} \quad p + p' \text{ in seconds} = \frac{r \sin. z + r' \sin. z'}{R \sin. 1''}.$$

$$\text{From I.} \quad (z + z') - (l + l') = \frac{r \sin. z + r' \sin. z'}{R \sin. 1''},$$

$$\text{and } R = \frac{r \sin. z + r' \sin. z'}{(z + z' - l - l') \sin. 1''};$$

but if the earth be considered a sphere, then $r = r'$,

$$\text{and } R = \frac{r\,(\sin. z + \sin. z')}{(z + z' - l - l') \sin. 1''}$$

$$= \frac{2r \sin. \frac{z + z'}{2} \cos. \frac{z - z'}{2}}{(z + z' - l - l') \sin. 1''},$$

a form adapted to logarithmic computation.

Now R being known,

$$p \text{ in seconds} = R \frac{r}{\sin. 1''} \sin. z,$$

$$\text{and } p' \text{ in seconds} = \frac{r'}{R.\sin. 1''} \sin. z'.$$

In practice the observatories need not be on the same meridian, because from the known change in the declination of an object it is very easy to reduce the zenith distance from the meridian where the observation is made to what it would be on another meridian near to it; and then the method investigated above can be applied. Stockholm, in longitude 18° 3′ 42″ E., in North latitude, and the Cape of Good Hope, in longitude 18° 28′ 42″ E., in South latitude, are particularly well situated for obtaining parallax by this method.

EXERCISE IV.

Ex. 89. What is meant by dip of the horizon? Show how to calculate it. Find from your formula the dip for 25, 26, 27 feet. *A.* 1869.

Ex. 90. Show by a figure how the correction for dip must be applied to the observed altitude.

Ex. 91. Prove the following rule for finding the distance of the horizon at sea is nearly correct. "Take the number of feet in the height of the eye above the sea, and increase it by half that number: the square root of this quantity will give the distance of the horizon in miles."

For Beaufort Testimonial, 1864.

Ex. 92. Neglecting the effect of refraction, calculate in degrees and minutes of arc the dip of the sea horizon as seen from a mountain 15,000 feet high.

B.A. and B.Sc. London, 1880.

Ex. 93. Explain the nature of refraction. Does it equally affect the positions of planets and fixed stars? Why?

Royal Naval College, 1868.

Ex. 94. Explain the phenomenon of the contraction of the moon's semidiameter: and investigate an expression for calculating the correction for it. *Honours,* 1882.

Ex. 95. Explain briefly how the state of the atmosphere, as indicated by the barometer and thermometer, may influence sextant observations. *Royal Naval College,* 1868.

Ex. 96. When are the corrections for temperature and barometric pressure not required for the altitude?

Ex. 97. State the laws of refraction, and prove that the amount varies as the tangent of the zenith distance.

Ex. 98. Prove the formula :—

$$\text{Refraction} = 57{\cdot}538'' \tan (z - 3\,r),$$

where z is the zenith distance, and r the mean refraction.

Ex. 99. Prove the formula for finding the refraction when the pressure and temperature of the atmosphere are both considered, viz. :—

$$\text{Refraction} = \frac{491{\cdot}1}{441{\cdot}1 + t^{\circ}} \times \frac{h}{29{\cdot}6} \times 57{\cdot}538''.\ \tan.\ (z - 3\,r),$$

where t is the temperature and h the height of the barometer.

Ex. 100. What is meant by the *augmentation of the moon's semidiameter;* what by the *contraction of the moon's semidiameter;* and in what cases are these corrections to be used?

Royal Naval College, 1864.

Ex. 101. The moon's equatorial diameter, taken out of the "Nautical Almanac," requires to be corrected for a given altitude: explain the reason for this. *Royal Naval College*, 1865.

Ex. 102. Given $p = 1640''$, $z' = 48^{\circ}\ 20'$, $R = 14'\ 15''$, find the augmentation of the moon's horizontal semidiameter from the formula :—

$$Aug. = 2R\,.\ \text{cosec.}\ (z' - p)\,.\ \sin.\ \tfrac{1}{2}p\,.\ \cos.\ (z' - \tfrac{1}{2}p).$$

Royal Naval College, 1866.

Ex. 103. Investigate a formula for finding the augmentation of the moon's horizontal semidiameter. Calculate the augmentation when the apparent altitude of the moon's centre is 40°, the horizontal parallax being 58′ 16″, and the horizontal semidiameter 15′ 54″. *Honours*, 1880.

Ex. 104. Prove the following formula for calculating the augmentation of the moon's horizontal semidiameter :—

$$Aug. = 2R\,.\ \text{cosec.}\ (z - p)\,.\ \sin.\ \tfrac{1}{2}p\,.\ \cos.\ (z - \tfrac{1}{2}p),$$

where p = hor. par. × cos. app. alt., and
R = horizontal semidiameter.

For Beaufort Testimonial, 1864.

Ex. 105. Find a, the augmentation of the moon's semidiameter, in the formula :—

$$a = As^2 \cos.\ z + \tfrac{1}{2}A^2 s^3 + \tfrac{1}{2}A^2 s^3 \cos.\ {}^2 z + \ldots$$

where $A = {\cdot}00001779 \ldots\ s = 942{\cdot}3''$, and
$z = 48^{\circ}\ 24'\ 10''$.

Royal Naval College, 1872.

Ex. 106. Supposing the greatest and least values of the

sun's apparent diameter to be 32′ 34·6″ and 31′ 30″; find the eccentricity of the earth's orbit. *B.A. and B.Sc. London*, 1874.

Ex. 107. What is meant by "correcting for the contraction of the moon's semidiameter" on account of refraction.

Royal Naval College, 1872.

Ex. 108. Investigate an expression for computing the contraction of the moon's semidiameter.

For Beaufort Testimonial, 1864.

Ex. 109. What two corrections are necessary to be made to the moon's equatorial horizontal parallax? Why?

Ex. 110. What do you mean by the "reduction of the moon's horizontal parallax?" Prove

$$h = H - H\frac{\sin.^2 l}{300},$$

where h is the horizontal parallax in latitude l, and H the equatorial horizontal parallax.

Ex. 111. Standing on the forecastle of a ship I observe two objects on shore, the nearer of which appears to the right of the other; but when observed from the poop they seem to have interchanged their position. Explain this, and state of which correction in Nautical Astronomy is it an illustration.

What is the difference between reflection and refraction? To what phenomena do the reflective and refractive powers of the atmosphere severally give rise. *A.* 1871.

Ex. 112. What are the effects of parallax and of refraction respectively on the apparent place of a body? Show by a figure the relative places on a circle of altitude (1) of the true and apparent sun, and (2) of the true and apparent moon. Obtain an expression for the parallax in altitude of a heavenly body, having given the horizontal parallax. *E.* 1872.

Ex. 113. What do you mean by the "correction in altitude" for the sun? When is the parallax of the sun greatest and when least? When is the refraction of the sun greatest and when least? Explain by diagrams. *E.* 1878.

Ex. 114. How may the moon's distance and magnitude be determined from her horizontal parallax and semidiameter?

Royal Naval College, 1864.

Ex. 115. Find the moon's distance from the earth when her horizontal parallax is 56′ 30″. *Royal Naval College*, 1864.

Ex. 116. Explain generally the meaning of the word parallax, and define *diurnal parallax*, *horizontal parallax*, and *parallax in altitude*.

If p be the equatorial horizontal parallax, P the parallax at a place whose latitude is l, and e the eccentricity of the earth, prove that

$$P = p\sqrt{1 - e^2 \sin.^2 l.}$$

Honours, 1869.

Ex. 117. Show by means of a figure that parallax depresses and refraction raises the altitude of a heavenly body. What is meant by the correction in altitude for refraction and parallax?

Royal Naval College, 1865.

Ex. 118. Investigate a formula for finding the moon's diurnal parallax, having given the true zenith distance.

The moon's horizontal parallax being 57′ 28·4″, and the true zenith distance 68° 34′ 30″, calculate the diurnal parallax.

Honours, 1881.

Ex. 119. The earth's radius being 3997 miles, and the moon's horizontal parallax 58′ 30″, what is the moon's distance from the earth's centre? *E.* 1869.

Ex. 120. Calculate the moon's parallax in altitude when her horizontal parallax = 56′ 15″, and her apparent altitude = 30°.

Royal Naval College, 1866.

Ex. 121. What do you mean by the *angle of the vertical*, and by *reduction for latitude?* Prove, the angle of the vertical varies as sine of twice the latitude, and hence find its greatest value.

Ex. 122. Define the *true* and *reduced* latitude of a spectator, *visible* and *rational* horizon, *latitude* and *longitude* of a heavenly body. *A.* 1868.

Ex. 123. Define and explain by diagrams *reduced zenith distance*, *horizontal parallax*, and *diurnal parallax.* Prove that diurnal par. = *hor. par.* × *sin. app. reduced Z D.*

Honours, 1879.

Ex. 124. Define *true* and *reduced* latitude of a place on the earth's surface, and investigate a formula for calculating the latter.

The true latitude is 48° 56′ N.; compute the reduced latitude, the earth's compression being $\frac{1}{313 \cdot 4}$. *Honours*, 1881.

Ex. 125. What corrections must be applied to an observed altitude of the sun to obtain the true altitude? Why is the observed place of the sun always above the true place? Obtain an expression for the correction for the dip of the horizon.

E. 1881.

Ex. 126. Find an expression for the diurnal parallax of a

heavenly body in terms of the true reduced zenith distance and the horizontal parallax. *Honours*, 1874.

Ex. 127. The parallaxes of the sun and moon being respectively 8·86″ and 57′, find approximately the ratio of the distances of the sun and moon from the earth.

Ex. 128. Having given the meridian zenith distances of a heavenly body observed at the same instant at two distant places on the same meridian on opposite sides of the equator, investigate a formula for the horizontal parallax.

Honours, 1874.

Ex. 129. Show that if α, β be the angles subtended between a star and the horizon at two heights a, b in the same vertical line, the radius of the earth $= \left\{ \dfrac{\sqrt{2b} - \sqrt{2a}}{\beta - \alpha} \right\}^2$ very nearly.

Ex. 130. The annual parallax of a star being 0·3136″, show that its distance is upwards of 600,000 times the distance of the earth from the sun (by circular measure).

For Beaufort Testimonial, 1865.

CHAPTER VI.

The "Nautical Almanac" and preliminary corrections—To correct the sun's declination – Example—Equation of time – How caused—Graphic representation of—To correct the equation of time—Example—To express intervals of mean time in sidereal time – Proof of formula used—Example—To correct the moon's right ascension and declination—Examples—To correct the moon's semidiameter and parallax —Example—On the correction of altitudes—Examples—Exercise—Examination.

THE "NAUTICAL ALMANAC" AND PREPARATORY CORRECTIONS.

"THE Nautical Almanac and Astronomical Ephemeris" is a work issued by the Admiralty for the special benefit of astronomers and navigators. It was first projected in 1767 by Dr. Nevil Maskelyne, the then Astronomer-Royal, and published with the authority of the Government. In 1834 it was altered to its present form, and such matter as the science of Astronomy then demanded was incorporated with it. It is published from three to four years in advance, to meet the wants of those who go on long voyages; and among other matter it contains all the data which is necessary (except those which can be obtained only by observation) for computing the latitude and longitude of the ship and the error of the compass, by giving the exact positions of the heavenly bodies at the times of their transits over the meridian of Greenwich. Now as these positions are constantly changing, it is necessary we should be able to reduce them to their exact values for all other times than those given. We shall now proceed to show how this is effected. As the "Nautical Almanac" is absolutely necessary in the calculations which follow, the student is urged to master its contents at his earliest convenience. Eighteen pages are allotted to each month for data relating to the sun and moon; pages 1 and 2 for each month contain the position of the sun at his transit at Greenwich for each day, page 1 for apparent noon, and page 2 for mean noon. The sun's semidiameter and the equa-

tion of time with its hourly variation, are also found on these pages. To facilitate the calculation of the position of the sun at any other time than noon, his variation in right ascension and declination *for one hour at noon* are given, and hence a Greenwich date is indispensable to compute the necessary reductions. The position of the observer cannot affect the absolute position of a heavenly body at any instant, and therefore, if its right ascension and declination be calculated for Greenwich, they are the same for the same time at every other place. It has been shown that the sun's position varies irregularly; hence we find the variation for one hour given in the "Nautical Almanac" differs from day to day. For the purposes of Nautical Astronomy no sensible error will be introduced by assuming the hourly variations to be constant throughout the day, because the errors from observation will almost universally be greater than the errors thus brought in. We shall in all cases correct the sun's position from the nearest noon, because less error is thus introduced; hence, for all times between noon and midnight, the hours, minutes, and seconds of the given date will be employed in the correction; but after midnight the hours, minutes, and seconds from the next noon will be used.

TO CORRECT THE SUN'S DECLINATION.

RULE (*a*) Reduce the time at place to the corresponding time at Greenwich. Find the hours, minutes, and seconds from the nearest noon, and reduce the result to hours and decimals of an hour.

(*b*) Take from the "Nautical Almanac" the declination for the nearest noon: from page 1 for apparent time, and from page 2 for mean time, and the variation for one hour from page 1.

(*c*) Multiply the variation for one hour by the number of hours and decimals of an hour from the nearest noon.

(*d*) This product must be applied to the declination at the nearest noon; to be added if the declination from the nearest noon to the Greenwich date be increasing, but to be subtracted if it be decreasing.

Ex. 131. Suppose it to be August 24th, 1887, at 10h. 47m. a.m. apparent time at place in longitude 83° 25′ W., what is the sun's correct declination?

For Greenwich time.					*Long. in time.*
App. time place, August	23d.	22h.	47m.	0s.	Long. 83° 25′ W.
Long., W.	∴ +	5	33	40	4
App. time Greenwich, Aug.	24	4	20	40	60)333 40
					5h. 33m. 40s.

Here the nearest noon is August 24th, and the number of hours from that noon are 4·34.

Sun's declination app. noon.		*Variation for time.*	
August 24	= 11° 8′ 11·9″ N.	Var. for 1h.	= − 51·33″
Correction	− 3 42·8	Time from noon =	4·34
True declination	= 11 4 29·1 N.		60)222·7722
		Correct. for 4·34h.	3′ 42·8″

Correct declination = 11° 4′ 29·1″ N.

Ex. 132. The mean time at ship in long. 127° 43′ 30″ E. was 1887, March 28d. 5h. 37m. 20s.; find the sun's correct declination.

For Greenwich time.					*Long. in time.*
Mean time at ship, March	28d.	5h.	37m.	20s.	Long. 127° 43′ 30″ E.
Long., E.	∴ −	8	30	54	4
Mean time, Greenwich, Mar. 27	21	6	26		60)510 54 0
					8h. 30m. 54s.

Hence time from noon of March 28d. = 2h. 53m. 34s.
= 2·89h.

Sun's declination mean noon.			*Variation for the time.*	
March 28	=	2° 58′ 37·2″ N.	Var. for 1h.	= 58·56″
Correction	= −	2 49·2	Time from noon =	2·89
True declination =		2 55 48 N.		60)169·2384
			Correction	2 49·2

Correct declination = 2° 55′ 48″ N.

EQUATION OF TIME.

The equation of time is the interval in mean time between apparent and mean times; and therefore serves for the conversion of either time into the other. It is measured by the angle at the pole between two hour-circles, one through the centre of

the true sun, and the other through the centre of the mean sun. As has been already explained, p. 34, the apparent sun has a variable motion from two causes: —

(1) Because the earth in her orbit travels faster in one part than it does in another. According to Keplar's second law the radius vector of each planet describes equal areas in equal times: hence when the sun is in perigee or the earth nearest the sun, for the area then described to be equal to the area described at any other time the arc of the orbit described must be greater than that other, and therefore the earth must then travel faster than at any other time; and when the distance from the sun is greatest or the sun in apogee, she must travel slower. The mean value of her motion will therefore be less than that in perigee and greater than that in apogee. Hence because of the earth's diurnal motion from W. to E. (which is the same as the sun's orbital motion), any meridian on the earth will pass through the mean sun before it passes through the true sun; that is, mean noon will occur before apparent noon between perigee and apogee. The reverse of this takes place from apogee to perigee; and therefore, from this cause (the unequal motion of the earth in her orbit) the equation of time is additive to apparent time from perigee to apogee, and subtractive from apogee to perigee, and at perigee and apogee the equation of time vanishes.

(2) Because the plane of the ecliptic is not at right angles to the axis of rotation, the sun's motion must appear at a maximum at the solstices and at a minimum at the equinoxes. Now, because the mean sun in the equinoctial travels with the mean motion of the true sun in the ecliptic, the two suns, if together at an equinox, will be together at every return to an equinox, and be on the same hour-circle at every return to a solstice. If we consider the two to start together at an equinox, the mean sun will at first outstrip the true sun, but they will come to the same hour-circle at the next solstice, and between these two points, a meridian on the earth by its diurnal rotation will pass through the apparent sun before the mean sun; that is, apparent noon will occur before mean noon; and hence from this cause the equation of time is subtractive from apparent time, when the sun is between an equinox and a solstice. By similar reasoning, when the sun is between a solstice and an equinox the equation of time is additive to apparent time, whilst at the equinoxes and solstices the equation of time

vanishes, because then the true and apparent suns are on the same hour-circle.

Hence, the equation of time is the algebraical sum of the parts of it due

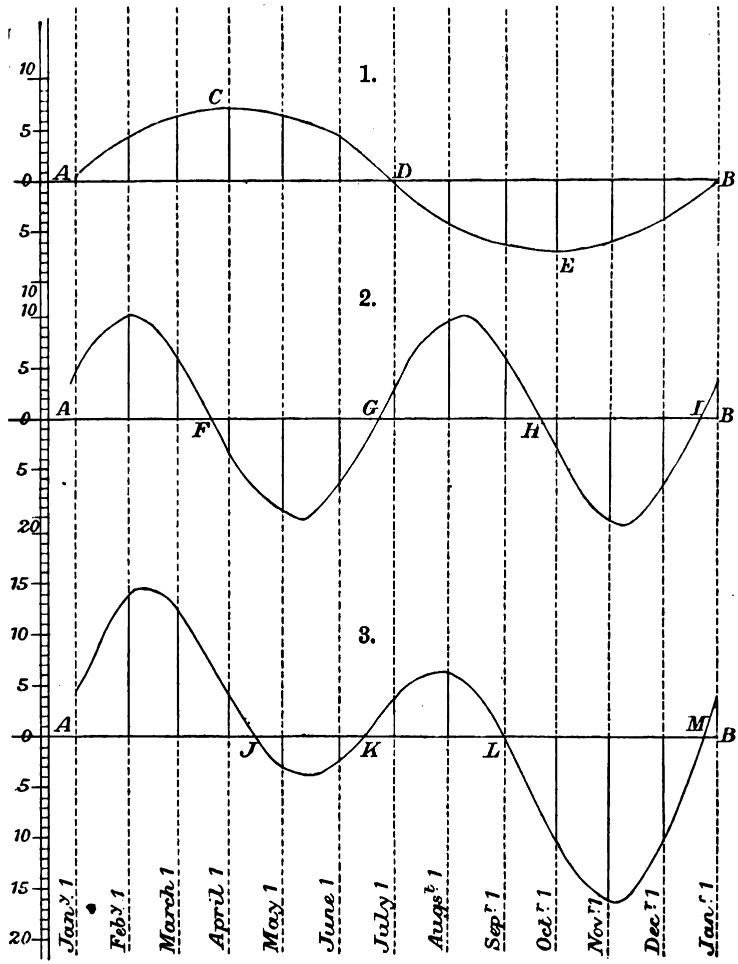
(*a*) To the unequal motion of the earth in her orbit, the maximum of this part being nearly seven minutes.

(*b*) To the obliquity of the ecliptic, the maximum of this part being nearly ten minutes; but the maxima from these causes of like signs do not occur at the same time, though nearly so, about November 1st, hence the equation of time never amounts to 17 minutes. These two causes produce a difference in the length of the apparent solar day of more than half an hour: the apparent noon in February taking place about $14\frac{1}{2}$ minutes after mean noon, and in November about $16\frac{1}{3}$ minutes before mean noon; and four times every year the equation of time vanishes. This may be shown graphically as follows:—

Take any straight line *A B* in all the figures to represent a year, and divide it into months. Then in Fig. 1 the curve *A C D E B* is drawn so that its ordinates—positive when above the line and negative below—shall represent that part of the equation of time due to the ellipticity of the earth's orbit. *A* and *B* represent the time (1st January) when the sun is in perigee, and *D* the time (1st July) when he is in apogee. The maximum ordinates are about 7 minutes taken from the scale on the left-hand, and, as already explained, is positive from perigee to apogee, and negative from apogee to perigee. In Fig. 2 the curve represents that part due to the obliquity of the ecliptic *A* and *I* denoting the time of the winter solstice, *F* that of the vernal equinox, *G* that of the summer solstice, and *H* that of the autumnal equinox. For reasons already explained the curve is positive from a solstice to an equinox, and negative from an equinox to a solstice, the longest ordinate being about 10 minutes taken from the scale. In Fig. 3 the algebraical sum of the ordinates is taken for the first day of every month, thus giving on these dates the total equation of time from both causes, and the curve is drawn as in the figure, where it may be seen the equation of time vanishes at *J K L M*, represented at about April 15th, June 15th, August 31st, and December 24th; because about these dates the algebraical sum of the positive and negative values of the ordinates in Fig. 1 and 2 become zero. The maximum values are about February

15th, when it is about + 14·5m.; May 14th, about − 3·9m.; July 25th, about + 6·2m.; and November 1st, about − 16·3m. Minute variations in the equation of time take place owing to the precession of the equinoxes, and to the progression of peri-

Fig 24.



gee in the direction of the sun's orbital motion, the former amounting, as already stated, to about 50″, and the latter to about 11″ annually, thus causing perigee to separate from the vernal equinox about 61″ every year.

To Correct the Equation of Time.

Like right ascension and declination, the equation of time is given for Greenwich apparent and mean noons, and hence in correcting it the same rules must be followed as were given for correcting the declination.

Ex. 133. 1887, February 14th, at 9h. 13m. 20s. a.m. apparent time at ship in longitude 107° 19′ 15″ W., what is the mean time?

For Greenwich date.

App. time at ship, Feb.	13d.	21h.	13m.	20s.
Long. in time. W. ∴	+	7	9	17
App. time Greenwich, Feb.	14	4	22	37

Long. in time.

Long.	107°	19′	15″ W.
			4
	60)429	17	0
	7h.	9m.	17s.

Equation of time.

February 14, app. noon	= +	13m.	25·61s.
Correction	—		1·64
True equation time	= +	13	23 97

Variation for time.

Var. 1h.	—	·375s.
Time from noon	4·38	
Cor. for 4·38h.		1·64250

Apparent time at ship, February		13d.	21h.	13m.	20s.
Correct equation of time	+			13	23·97
Mean time at ship, February		13	21	26	43·97

Having seen how to find the equation of time for any given date, we shall now proceed to show how an interval of mean time may be converted into one of apparent time, and here the student must be cautioned never to confound an interval with a date. *An interval* may be defined as the portion of time elapsed between any two events; but *a date* is a moment of absolute time, or the interval elapsed since some great event considered as a standard, the absolute instant when that happened being reckoned as zero. Thus 5 hours' interval of mean time is very different from 5 hours' mean time on a given day. The former means any five hours of mean time taken from any part of the day; the latter means the 5th hour of mean time on a certain day of a given month and year.

Because the apparent solar days alter in length according to the time of year, whilst mean solar days remain of uniform length, so we shall find an interval of apparent solar time will differ from an interval of mean solar time at different times of the year. When the apparent solar day is shorter than the mean solar day, the apparent solar hour and its divisions must be shorter than the mean solar hour and its divisions; and hence in any absolute interval there will be a greater number of apparent solar hours than of mean solar hours; and then to convert an interval of mean solar time into its equivalent interval of apparent solar time the proportional part of the equation of time must be applied. Now when the equation of time is additive and increasing, or subtractive and decreasing from mean time, then 24 hours of mean time are greater than 24 hours of apparent time, and the apparent solar day is shorter than the mean solar day; and is so by as much as the equation of time changes in that day. But when the equation of time is additive and decreasing, or subtractive and increasing, then 24 hours of mean time are less than 24 hours of apparent time, and the apparent solar day is longer than the mean solar day by the change in the equation of time for that day.

The proof of this statement is as follows:—

Let $\tau =$ the absolute *mean time* of first observation.

$\epsilon =$ the equation of time to be applied to the mean time of first observation.

$\therefore\ \tau \pm \epsilon =$ the absolute *apparent time* of first observation.

Again,—

Let $\tau' =$ the absolute *mean time* of second observation; and $\epsilon \pm \delta\epsilon =$ the equation of time to be applied to the mean time of second observation.

$\therefore\ \tau' \pm (\epsilon \pm \delta\epsilon) =$ the absolute *apparent time* of second observation.

Hence,—

$\{\tau' \pm (\epsilon \pm \delta\epsilon)\} - (\tau \pm \epsilon) =$ interval of apparent time between the observations;

that is, apparent interval $= \tau' - \tau + \delta\epsilon$.

Let us now discuss the different signs:—

(a) If the equation of time be *additive* to mean time and *increasing*, we get

$$\text{Apparent interval} = \{\tau' + (\epsilon + \delta\epsilon)\} - (\tau + \epsilon)$$
$$= \tau' - \tau + \delta\epsilon \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(β) If the equation of time be *subtractive* from mean time and *decreasing*, then

$$\text{Apparent interval} = \{\tau' - (\epsilon - \delta\epsilon)\} - (\tau - \epsilon)$$
$$= \tau' - \tau + \delta\epsilon \quad . \quad . \quad . \quad . \quad . \quad (2)$$

(γ) If the equation of time be *additive* to mean time and *decreasing*, we get

$$\text{Apparent interval} = \{\tau' + (\epsilon - \delta\epsilon)\} - (\tau + \epsilon)$$
$$= \tau' - \tau - \delta\epsilon \quad . \quad . \quad . \quad . \quad . \quad (3)$$

(δ) If the equation of time be subtractive from mean time and *increasing*, we have

$$\text{Apparent interval} = \{\tau' - (\epsilon + \delta\epsilon)\} - (\tau - \epsilon)$$
$$= \tau' - \tau - \delta\epsilon \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In each of the above four cases $\tau' - \tau$ is the interval between the observations in mean time, and $\delta\epsilon$ the part of the equation of time due to the elapsed interval; hence the rule as previously stated for converting an interval of mean time into its corresponding interval of apparent time. The rule to be followed is,—

RULE. Multiply the variation of the equation of time in one hour from page 1, "Nautical Almanac," by the number of hours and decimals of an hour in the given interval; then

(*a*) If the equation of time, page 2, "Nautical Almanac," be additive and increasing or subtractive and decreasing, the result must be added to the mean interval; but

(*b*) If the equation of time, page 2, be additive and decreasing or subtractive and increasing, the result must be subtracted from the mean interval to obtain the apparent interval.

Ex. 134. Reduce an interval of 5h. 46m. 30s. of mean solar time on July 11th, 1887, to an interval of apparent solar time, and to its equivalent interval in sidereal time. Also reduce, July 11th, 1887, 5h. 46m. 30s. of mean solar time to its equivalent apparent solar time, the longitude being 73° 20′ E.

$$
\begin{array}{rr}
\text{5h. 46m. 30s.} = & 5{\cdot}775\text{h.} \\
\text{Var. of E. T. in 1h.} = & \underline{{\cdot}337} \\
 & 40425 \\
 & 17325 \\
 & 17325 \\
\text{Var. of E. T. in } 5{\cdot}775\text{h.} = & \overline{1{\cdot}946175}
\end{array}
$$

Now on July 11th the equation of time is subtractive from mean time and it is increasing: hence the above result must be subtracted from the mean interval.

Interval in mean time	=	5h.	46m.	30s.
Correction for E. T.	=		–	1·95
∴ Interval in apparent time	=	5	46	28·05

Again, to obtain the sidereal interval, turn to the table of time equivalents, page 480.

5h. mean interval	=	5h.	0m.	49·2824	sid. interval.
46m. ,,	=		46	7·5566	,,
30s. ,,	=			30·0821	,,
∴ 5h. 46m. 30s. mean interval	=	5	47	26·9211	sid. interval.

Lastly, to find the equivalent apparent solar time—

For Greenwich date.

M. T. at place, July	11d.	5h.	46m.	30s.
Long., E. ∴	–	4	53	20
M. T. at Greenwich, July	11	0	53	10

Long. in time.

Long. 73° 20′ E.
4

60)293 20

4h. 53m. 20s.

Equation of time.

July 11	=	5m.	10·01s.
Correction	=	+	·30
		– 5	10·31

Variation for time.

Variation for 1h.	+ ·337
Number of hours	·886
	·298582

Mean time at place, July	11d.	5h.	46m.	30s.
Equation of time		–	5	10·31
Apparent time at place, July	11	5	41	19·69

To Correct the Moon's Right Ascension and Declination.

The moon's motion in the heavens is very much more complex than the sun's, and is easier to be understood when referred to the ecliptic than when referred to the equinoctial. Her path is inclined to the sun's at an angle of about 5° 9′, but she does not describe accurately (though nearly) a great circle in the celestial concave. If one be drawn through the direction of the moon's motion at any instant, it will cut the ecliptic in

two opposite points, called *the moon's nodes*, at an angle, as before stated, of 5° 9′ nearly. At every revolution of the moon round the earth there is a regression of the moon's nodes in the ecliptic precisely analogous to the precession of the equinoxes, due to the disturbing action of the sun, amounting to 1° 27′ in each sidereal revolution of the moon. In one year the nodes are therefore carried back 19° 20′, and thus it takes 18·6 years for them to pass through the whole ecliptic and to regain their former positions. It is only after this cycle the moon revolves through the same path, giving rise to similar phenomena as it did before. The greatest latitude will therefore be always 5° 9′; but from the position of the line of nodes her declination will vary from about 23° 27′ + 5° 9′, where the plane of her orbit reaches nearer the pole to about 23° 27′ − 5° 9′, where the plane of her orbit is further removed from the pole than that of the ecliptic. This causes very large variations in her right ascension and declination from day to day, in addition to which her angular motion in her orbit is so great (being about thirteen times as great as the sun's), that it has been found convenient to tabulate her right ascension and declination in the "Nautical Almanac" for every hour of the day. This is done at pages from 5 to 12, whilst the variations of these co-ordinates are given for every ten minutes. The corrections of these are made precisely similar to those for the sun, except in that they must be taken for the next less hour of the Greenwich mean time and corrected for the number and parts of ten minutes from that next less hour.

Ex. 135. 1887, June 27th, at 9h. 40m. p.m. mean time at ship in long. 80° 20′ E., find the moon's right ascension and declination.

For Greenwich date.

M. T. at place, June	27d.	9h.	40m.	0s.
Long., E.	−	5	21	20
M. T. Greenwich, June	27	4	18	40

Long. in time.

80° 20′
4
60)321 20
5h. 21m. 20s.

For Moon's R. A.

June 27,	4h. =	11h.	49m.	2·53s.
Correction	+			41·78
Corrected R. A. =		11	49	44·31

Variation in 10m.

	+ 22·377s.
No. of 10m. =	1·867
	41·777859

For Moon's declination.		*Variation in* 10*m.*
June 27, 4h.	= 4° 29′ 34·6″ N.	− 117·02″
Correction	− 3 38·5	1·867
Corrected dec.	= 4 25 56·1 N.	60)218·47634
		3′ 38·5″

To Correct the Moon's Semidiameter.

The corrections necessary for the semidiameter of the moon have been already fully investigated at pages 48—50, and there shown that the augmentation of the moon's semidiameter is obtained from the formula,

$$\text{Augmentation} = 2s \,.\, \cos.\ \frac{2z - p}{2} \,.\, \sin.\ \frac{p}{2} \,.\, \text{cosec.}\ (z - p),$$

and all that is now necessary is to show how this is practically used.

RULE (*a*) Obtain the Greenwich mean time of observation.

(*b*) From the "Nautical Almanac," page 3, take out the semidiameter for the noon and midnight between which the Greenwich date lies.

(*c*) The difference between these is that for 12 hours; hence, multiply the difference by the number of hours and decimals of an hour between noon or midnight and the Greenwich time, and divide the product by 12 for the proportional part of the correction.

(*d*) The result must be added to the semidiameter if it be increasing, but subtracted if decreasing.

(*e*) From the nautical tables (Riddle, p. 3) take out the augmentation of the moon's semidiameter calculated from the above formula and there tabulated: and add to the last result (*d*). This gives the correct semidiameter.

Ex. 136. 1887, June 11th, at 2h. 27m. a.m. at a ship in long. 30° 10′ E., find the moon's correct semidiameter if her true altitude be 58° 20′.

For Greenwich date.					*Long. in time.*
Time at ship, June	10d.	14h.	27m.	0s.	30° 10′ E.
Long. in time, E.	−	2	0	40	4
Time at Greenwich, June	10	12	26	20	2h. 0m. 40s.

Correction for semidiameter.		
At midnight	10d.	$= 15' \ 5{\cdot}0''$
At noon	11	$= 15 \ \ 0{\cdot}7$
Difference in 12h.		$= - \ 4{\cdot}3$
	26m. 20s.	$= \ {\cdot}439$
		$12)1{\cdot}8877$
Difference in 26m. 20s.		$= -{\cdot}1573$

Semidiameter.	
At midnight	$= 15' \ 5{\cdot}0''$
Correct. for 26m. 20s.	$= - \ \ {\cdot}16$
	$= 15 \ \ 4{\cdot}84$
Augmentation	$+ 12{\cdot}33$
Correct semidiameter	$= 15 \ 17{\cdot}17$

To Correct the Moon's Parallax.

As shown in Chapter V., pages 56—61, the moon's horizontal parallax as deduced from the "Nautical Almanac," must undergo two corrections before it can be used in calculations, because what is tabulated there is *the equatorial horizontal parallax.* These corrections are :—

(1) For the latitude of the place called the reduction of the moon's equatorial horizontal parallax from the formula—

$$h = H - H\frac{a - b}{b}\sin.^2 l.$$

The right-hand member is tabulated in all good nautical tables (Riddle, p. 3).

(2) For the moon's altitude from the formula—

Par. in alt. = hor. par. + cos. app. alt.

We have now to show how this is performed in practice.

RULE (*a*) Obtain the Greenwich mean time and correct the moon's equatorial horizontal parallax similar to that of the semidiameter, using the table for the reduction of parallax instead of that for the augmentation of the semidiameter.

(*b*) Reduce the corrected horizontal parallax to seconds, and add together the log. cosine of the apparent altitude and the log. of the number of seconds in the corrected horizontal parallax; the result is the log. of the number of seconds in the parallax in altitude.

(*c*) From the apparent altitude subtract the refraction and add the parallax in altitude; the result is the true altitude.

Ex. 137. 1887, July 7th, at 9h. 22m. p.m. mean time at ship in latitude 48° 23′ S., longitude 164° 23′ 30″ W., find the moon's true altitude if her apparent altitude be 18° 28′ 52″.

For Greenwich date.				*Long. in time.*
Mean time ship, July 7	9h.	22m.	0s.	Long. 164° 23′ 30″ W.
Longitude, W.	+ 10	57	34	4
M. T. Greenwich, July 7	+ 20	19	34	60)657 34 0
				10h. 57m. 34s.

Correction for horizontal parallax.		*Horizontal parallax.*	
July 7, at midnight	55′ 34·6″	. . . . .	55′ 34·6″
„ 8, at noon	55 18·6	Cor. for 8h. 19m. 34s.	− 11·1
Difference for 12h.	− 16		55 23·5
No. of hrs.	8·326	Reduction	− 6·2
	12)133·216	True horizon. parallax	55 17·3
			= 3317·3″
Cor. for 8h. 19m. 34s.	= 11·1		

To correct the altitude.			
App. alt.	18° 28′ 52″	. . . . . .	Cos. 9·977005
Refraction	− 2 49		
	18 26 3	. Horizontal parallax 3317·3	log. 3·520784
Parallax in alt.	+ 52 26 =	3146·	log. 3·497789
True alt.	19 18 29		

On the Correction of Altitudes.

As has been already stated, the altitudes of objects when observed at sea must be corrected to reduce them to the true geocentric altitudes. The corrections employed are for index error, height of the eye, semidiameter of the object, refraction and parallax: and the manner of applying these corrections is easily understood from the following rule in the case of the sun.

Rule (*a*) Apply the *index error* to the observed altitude according to its sign.

(*b*) Take from the nautical tables used the *dip for the height of the eye* and subtract.

(*c*) Apply the *semidiameter* from the "Nautical Almanac," + when the lower limb is observed and − when the upper limb is observed; the result is the apparent altitude.

(*d*) From the nautical tables take the *refraction* for the altitude of the object as found in (*b*). This is always subtractive.

(*e*) Add the *parallax* from the tables for the altitude of the sun.

The final result is the true altitude of the sun.

When the artificial horizon is used there will be no correction for dip, and the altitude after being corrected for index error must be halved before the other corrections are applied. This has been already explained when treating of the artificial horizon.

Ex. 138. 1887, September 23rd, the observed altitude of the sun's L. L. was 48° 23′ 30″. The height of the eye 22 feet. Index error + 2′ 17″. Find the true altitude.

Observed altitude ☉	48° 23′ 30″
Index error	+ 2 17
	48 25 47
Dip for 22 feet	− 4 37
	48 21 10
Semidiameter	+ 15 59
	48 37 9
Refraction	− 50
	48 36 19
Parallax	+ 6
True altitude	48 36 25

Ex. 1877, October 17th, the observed altitude of the sun's U. L. in an artificial horizon was 121° 13′ 20″. Height of eye 27 feet. Index error − 1′ 54″. Find the true altitude.

Observed double altitude	121° 13′ 20″
Index error	− 1 54
	2)121 11 26
	60 35 43
Semidiameter	− 16 6
	60 19 37
Refraction	− 33
	60 19 4
Parallax	+ 4
True altitude	60 19 8

The discs of stars are so small that their semidiameters do not admit of measurement, and their distances are so great

that at their surfaces the semidiameter of the earth subtends no appreciable angle; hence for stars there is no correction for either semidiameter or parallax, but the other corrections remain the same as for the sun.

Ex. 139. 1887, April 17th, the observed altitude of α *Centauri* was 23° 40′. Height of the eye 25 feet. Index error − 2′ 13″. Find the true altitude.

Observed altitude *	23° 40′ 0″
Index error	− 2 13
	23 37 47
Dip for 25 feet	− 4 55
	23 32 52
Refraction	− 2 11
True altitude *	23 30 41

In the case where the moon or one of the nearer planets is observed, its semidiameter must be augmented as has been already explained before it can be applied; and its parallax from the "Nautical Almanac" must be corrected from the formula par. in alt. = hor. par. + cos. app. alt. This is illustrated in the next two examples.

Ex. 140. The observed altitude of the L. L. of the planet Venus on 20th December, 1887, was 57° 29′ 45″. The index error of the sextant was + 14′ 17″, height of the eye 37 feet. The artificial horizon was used and the semidiameter and the horizontal parallax taken from the "Nautical Almanac" were 10·1″ and 10·8″ respectively. Find the true altitude.

Observed double alt.	57° 29′ 45″	
Index error	+ 14 17	
	2)57 44 2	
	28 52 1	
Semidiameter	+ 10	
Apparent altitude	28 52 11	. . . . . Cos. 9·942365
Refraction	− 1 43	
	28 50 28	Horizontal parallax 10·8 by 1·033424
Parallax in alt.	+ 9·5	9·5 by 0·975789
True altitude	28 50 37·5	

The correction of the moon's altitude requires a little more care, and it will be well for the student to follow the arrangement here laid down.

Ex. 141. 1887, November 20th, at 3h. 13m. 40s. p.m. mean time at ship in latitude 41° 56′ S., longitude 38° 49′ E., find the moon's true altitude if the observed altitude of her lower limb be 59° 54′ 40″. The index error of the sextant + 5′ 15″. Height of eye above the sea 22 feet.

For Greenwich date.

Mean time ship, Nov. 20	3h. 13m. 10s.
Long., E.	− 2 35 16
Greenwich M. T. Nov. 20	0 38 54

Long. in time.

Long. 38° 49′ E.
4
155 16
2h. 35m. 16s.

Correction for moon's semidiameter.

At noon. Nov. 20 =	15′ 29·2″
At midnight. „	15 22·2
Difference in 12 hrs.	− 7·
	·64h.
	12)4·48
Correction for 38m. 54s.	− ·37

For horizontal parallax.

56′ 44·4″
56 18·7
− 25·7
·64
1028
1542
12)16·448
1·37

For moon's semidiameter.

At noon, November 20 =	15′ 29·2″
Correction for 38m. 24s.	− 4
	15 28·8
Augmentation	+ 13·4
Corrected semidiameter	15 42·2

For horizontal parallax.

	56′ 44·4″
	− 1·4
	56 43
Reduction	− 5
Correct H. P.	56 38
	= 3398″

To correct the altitude.

Observed alt. moon's L. L.	59° 54′ 40″	
Index error	+ 5 15	
	59 59 55	
Dip for 22 feet	− 4 37	
	59 55 18	
Semidiameter	+ 15 42	
Apparent altitude	60 11 0	. . Cos. 9·696554
Refraction	− 33	
	60 10 27	H. P. 3398 log. 3·531223
Par. in alt.	+ 28 10 =	1690 log. 3·527777
True altitude, moon's centre	60 38 37	

EXERCISE V.

Ex. 142. 1887, September 1st, at 3h. 27m. 19s. mean time at ship in longitude 145° 15′ W., find the Greenwich time and correct the sun's declination and equation of time.

Ex. 143. 1887, March 21st, 9h. 15m. 47s. a.m. Apparent time at ship in longitude 173° 28′ 30″ E., find the Greenwich time and the sun's declination and equation of time.

Ex. 144. 1887, August 14th, 10h. 40m. a.m. mean time at place in longitude 93° 20′ 45″ W., find the true right ascension and declination of the moon and her correct semidiameter and parallax in altitude if the latitude be 27° 20′ N. and apparent altitude 51° 41′ 43″.

Ex. 145. Why do not the sun's and moon's declinations and right ascensions change uniformly? Explain fully with diagram.

Ex. 146. 1887, May 25th, the observed altitude of the sun's L.L. was 27° 13′ 45″. Index error + 18″. Height of eye 21 feet. Find the true altitude of the sun.

Ex. 147. 1887, February 27th, with an artificial horizon the observed altitude of the sun's upper limb was 83° 27′ 10″. Height of the eye 24 feet. Index error of the sextant — 7′ 21″. Required the true altitude.

Ex. 148. 1887, November 19th, with an artificial horizon the observed altitude of the star *Rigel* was 117° 20′. Height of the eye 23 feet. Index error + 4′ 13″. Find the true altitude.

Ex. 149. 1887, March 17th, the observed altitude of the planet Jupiter's L. L. was 53° 12′ 20″. Height of the eye 19 feet. Index error — 4′ 23″. Find the true altitude.

Ex. 150. 1887, February 13th at 8h. 17m. 16s. a.m. mean time at ship in latitude 12° 18′ N., longitude 126° 47′ W., the observed altitude of the moon's lower limb was 33° 43′. The index correction for the sextant + 3′ 4″. Height of the eye 19 feet. Find the true altitude of the moon's centre.

Ex. 151. 1887, October 3rd at 4h. 37m. a.m. mean time at ship in latitude 14° 39′ S., longitude 42° 51′ E., the observed altitude of the moon's upper limb was 26° 30′ 30″. Height of eye 23 feet. Index error — 2′ 59″. Find the true altitude of the moon's centre.

Ex. 152. What is the "Nautical Almanac"? Why is it so necessary for astronomical calculations? Explain clearly why the variation in the sun's co-ordinates are given for one hour, and for the moon for ten minutes.

Ex. 153. In all minor corrections of data taken from the "Nautical Almanac" a Greenwich date is required. Why is this necessary?

Ex. 154. Show clearly how a date in mean time may be converted into its equivalent in apparent time, and also how an interval in mean time may be converted into its equivalent interval of apparent time.

Ex. 155. What are the limits of the moon's declination? Show by a diagram and explain fully the reason for your answer.

Ex. 156. Take out of the "Nautical Almanac" the sun's declination and the equation of time; (1) for a place in longitude 47° 50′ W. on May 8th, 1887, at 6h. 40m. a.m. mean time; (2) for a place in longitude 124° E. on November 5th, 1887, at 3h. 30m. p.m. mean time. *E.* 1881.

Ex. 157. Define equator, first meridian, horizon, right ascension and declination. Explain your definitions by diagrams.

In the "Nautical Almanac" under March 20th—21st, 1878, we find the declination of the sun given respectively as S. 0° 5′ 38·7″ and N. 0° 18′ 2·4″; explain clearly what these letters and figures indicate as to the position of the sun.

E. 1878.

Ex. 158. Explain the nature of the correction called "equation of time;" show with figure, why it is sometimes additive and sometimes subtractive. Why does it sometimes vanish?

Royal Naval College, 1872.

Ex. 159. What is meant by "the equation of time"? To what causes is it due? If h be the time between sunrise and noon, and h' between noon and sunset, show that the equation of time $= \dfrac{h \backsim h'}{2}$.

On the 10th January, 1881, the sun rose in London at 8h. 6m. and set at 4h. 10m. What was the equation of time? Was it additive or subtractive from mean time? *A.* 1881.

Ex. 160. Explain the construction and use of the table in the "Nautical Almanac" headed "equation of time."

Royal Naval College, 1867.

Ex. 161. What is the nature of the observed path of the sun amongst the fixed stars? May this apparent motion be due to a real motion of the earth? Explain why. Why do we infer that the sun's distance from the earth varies through-

out the year? Does the sun move uniformly amongst the fixed stars? *For B.A. London University*, 1854.

Ex. 162. Define "equation of time," and explain fully the causes of it. Show that the equation of time vanishes four times a year. *Honours*, 1873.

Ex. 163. How is the moon's semidiameter taken out of the "Nautical Almanac" for a Greenwich date?

August 18th, 1887, at 5h. 30m. a.m. in a place whose longitude is 45° 30′ E., required the moon's semidiameter.

E. 1868.

Ex. 164. The moon's equatorial diameter taken out of the "Nautical Almanac" requires to be corrected for a given altitude; explain the reason for this.

Royal Naval College, 1865.

Ex. 165. Explain in what manner quantities taken from the tables in the "Nautical Almanac" headed "moon's semidiameter" and "moon's horizontal parallax" are corrected when made use of. *Royal Naval College*, 1867.

CHAPTER VII.

Hour angles and meridian passages—Proof of formula—To find the hour angle of a heavenly body—To find what stars are near the meridian—To find at what time a star will cross the meridian—The moon's meridian passage—To find the Greenwich date of the moon's meridian passage—Examples—Exercise—Examination.

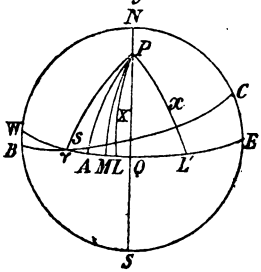
Hour Angles and Meridian Passages.

The hour angle of a heavenly body is the angle at the pole, included between the hour circle through the centre of the body and the meridian of the observer.

In the case of the sun, when it is apparent noon the centre of the true sun is on the meridian of the place of observation, and the number of degrees, &c., he is east or west of the meridian at any instant measured at the pole and expressed in time, is the apparent interval from noon: hence his hour angle, if he be west of the meridian, is the apparent time at place; but if he be east of the meridian, it is 24 hours minus the apparent time at place.

In the case of every other heavenly body, the sidereal time and the right ascension of the body under consideration must be taken into account. Bearing in mind that the sidereal time at any place is the interval in mean time elapsed since the "first point of Aries" crossed the meridian, that is, the right ascension of the meridian, and that the heading in the "Nautical Almanac," page 2 of each month, "*Sidereal time at mean noon,*" is the right ascension of the first meridian at mean noon at Greenwich, or is "the angular distance of the first point of Aries, or the true vernal equinox from the meridian at the instant of mean noon" at Greenwich, we may proceed to show how the hour angles of all heavenly bodies are found.

Fig 25.



Let fig. 25 represent the projection of the celestial concave on the plane of the horizon, where

NS is the meridian of the observer,
WQE „ „ equinoctial,
$B \Upsilon C$ „ „ ecliptic,
then Υ where the two latter intersect is the first point of Aries.

Let s be the position of the true sun in the ecliptic, and M be the position of the mean sun in the equinoctial, draw circles of declination $P\Upsilon$, PsA and PM, then

ΥPQ is the right ascension of the meridian,
ΥPM „ „ mean sun,
ΥPA „ „ true sun,

and QPM the angular distance of the mean sun from the meridian is the measure of mean time.

QPA the angular distance of the true sun from the meridian is the measure of apparent time.

MPA the angular distance of the mean sun from the true sun is the measure of the equation of time.

$QP\Upsilon$ the angular distance of the first point of Aries from the meridian is the measure of sidereal time.

Then we have—

R. A. of meridian $QP\Upsilon = \Upsilon PM + QPM$

$\therefore$ Sidereal time = R. A. of mean sun + mean time. I.

or mean time = R. A. of meridian − R. A. of mean sun.

Now if X be the position of any body, draw the declination circle PXL, then QPL is its westerly hour angle, ΥPL its right ascension, and we have

$$\Upsilon PQ = \Upsilon PL + QPL.$$

that is—R. A. of meridian or sidereal time = R. A. of the body + its westerly hour angle . . . . II.

and, westerly hour angle = R. A. of meridian − R. A. of the body . . . . III.

Substituting the value of R. A. of the meridian found in I., westerly hour angle = R. A. of mean sun + mean time at place − R. A. of the body . . . . IV.

and mean time = westerly hour angle + R. A. of the body − R. A. of the mean sun . . . . V.

If x denote the object east of the meridian, we get

$$QPL' = \Upsilon PL' - \Upsilon PQ.$$

∴ Easterly hour angle = R. A. of the body − R. A. of meridian . . . . VI.

but the westerly hour angle is 24 hours − easterly hour angle.

$$\therefore 24\text{h.} - QPL = 24\text{h.} + \Upsilon PQ - \Upsilon PL',$$

or westerly hour angle = 24 h. + R.A. of meridian − R. A. of the body . . . . VII.

Ex. 166. Find the R. A. of the mean sun and R. A. of the meridian for 1887, October 29th, 17h. 35m. 42s. mean time at Greenwich.

R. A. meridian = R. A. mean sun + mean time.

Sidereal time at mean noon 29th October (p. 2, N. A.)

		=14h.	29m.	59·91s.
From table of time equivalents Acceleration for	17h. . .		2	47·56
	35m. . .			5·75
	42s. . .			·11
Right ascension of the mean sun		14	32	53·33
Mean time at place		17	35	42
Right ascension of the meridian		8	8	35·33

TO FIND THE HOUR ANGLE OF A HEAVENLY BODY.—In equation IV. it was shown that the westerly meridian distance or hour angle is obtained from R. A. of body

h = R. A. mean sun + mean time—R. A. of body.

Ex. 167. The mean time at Plymouth is January 15th, 1887, 7h. 23m. 19s, and longitude 4° 7′ 16·5″ W., find the hour angle of Polaris.

For Greenwich time.

Mean time at place, Jan.	15d.	7h.	23m.	19s.
Long. in time, W.	+		16	29·1
Mean time Greenwich, Jan.	15	7	39	48·1

Long. in time.

4°	7′	16·5″ W.
		4
16	29	6

R. A. of mean sun.

Sidereal time N. A.	19h.	38m.	28 88s.
Acceleration for 7h.		1	9·00
39m.			6·41
48·1s.			·13
R. A. mean sun	19	39	44·42

Hour angle.

R. A. mean sun	19h.	39m.	44·42s.
Mean time place	7	23	19
R. A. meridian	27	3	3·42
R.A. Polaris (N.A.)	1	17	35 47
Omitting 24h.	1	45	27·95
	60		
	4)105	27·95	
Hour angle	26°	21′	59·25″

TO FIND WHAT STARS ARE NEAR ANY MERIDIAN.—When a star is on the meridian it has no hour angle, and then

R. A. mean sun + mean time at place − R. A. of star = 0,

or R. A. mean sun + mean time at place } i.e. right ascension of meridian } = R. A. of star,

from which the following rule is deduced.

To find what bright stars are near the meridian.

RULE (*a*) Find Greenwich mean time.

(*b*) Find the R. A. of the mean sun.

(*c*) To the R. A. of the mean sun, add the mean time at place; the sum is the R. A. of the meridian, and the stars which have nearly the same R. A. are those required.

Ex. 168. 1887, May 10th, 8h. 10m. 40s. p.m. mean time at place in longitude 86° 10′ 12″ E., what bright stars are near the meridian?

For Greenwich date.

Mean time at place, May	10d.	8h.	10m.	40s.
Long. in time, E.	−	5	44	40·8
G. M. time, May	10	2	25	59·2

Long. in time.

86°	10′	12″
		4
60)344	40	48
5	44	40·8

R. A. of mean sun.

Sidereal time, May 10 =	3h.	11m.	52·51s.
Acceleration for 2h.			19·71
Acceleration for 25m.			4·11
Acceleration for 59s.			·16
R. A. mean sun	3	12	16·49

R. A. of meridian.

R. A. mean sun	3h.	12m.	16·49s.
M. T. place	8	10	40
R.A. meridian	11	22	56·49

By comparing the R. A. of the meridian with the R. A. of the fixed stars in the "Nautical Almanac," we find τ *Leonis* a little to the west of the meridian, and λ *Draconis* a little to the east of the meridian, and these are nearest to the meridian.

TO FIND THE TIME WHEN ANY PARTICULAR STAR WILL CROSS THE MERIDIAN.—It has been already shown that a star has no hour angle when it is on the meridian, and then

R. A. mean sun + mean time at place − R. A. of star = 0;

∴ mean time at place = R. A. of star − R. A. of mean sun.

To find the time of a star's meridian passage.

RULE (*a*) Find the approximate mean time of transit by subtracting the R. A. of the mean sun at mean noon from the R. A. of the star.

(*b*) With this approximate mean time of transit, find the Greenwich mean time and the correct R. A. of the mean sun. Subtract the latter from the R. A. of the star, the remainder is the mean time of transit. Where great accuracy is required this may be repeated.

Ex. 169. At what time will *α* Hydræ pass the meridian of a place in 125° 14′ W., on May 24th, 1887?

R. A. of *α* Hydræ (N. A.)	9h.	22m.	2·52s.
R. A. mean sun (N. A.)	4	7	4·29
Approximate mean time transit	5	14	8·23

For Greenwich date.

Mean time at place, May 24		5h.	14m.	58·23s.
Long. in time, W.	+	8	20	56
G. M. time, May 24		13	35	54·23

Long. in time.

125° 14′ W.
4

60)500 56

8h. 20m. 56s.

R. A. mean sun.

Sidereal time, May 24 =	4h.	7m.	4·29s.
Acceleration for 13h.		2	8·13
Acceleration for 35m.			5·75
Acceleration for 54s.			·15
R. A. of mean sun	4	9	18·32

Meridian passage of star.

R. A. *α* Hydræ	9h.	22m.	2·52s.
R.A. mean sun	4	9	18·32
M. T. transit	5	12	44·20

SIDEREAL TIME.—Sidereal time is the same as the right ascension of the meridian, and is, therefore, found in the same manner from the formula—

Sidereal time = R. A. of mean sun + mean time at place.

THE MOON'S MERIDIAN PASSAGE.—By this term in the "Nautical Almanac," page IV for each month, is meant the mean time at which the moon's centre is on the upper meridian at Greenwich. By looking down the column a symbol ** is seen, which indicates that there is no passage of the moon across the upper meridian of Greenwich on that day. This is the case "once in every lunation, and arises from the circumstance of the lunar day being greater than the mean solar day, and including it within its limits." From the moon's proper motion to the eastward, her transit across any given meridian must take place later each day than

the preceding one, and the interval it is later (variable in its duration), is called her *daily retardation;* and is found by subtracting the time of the moon's meridian passage on any day from that following. For example, the moon passed the meridian of Greenwich at 0h. 46·3m. on 25th March, 1887, and at 1h. 28·9m. on 26th March: the difference between these times of transit is 24h. 42·6m.: then 42·6m. is the daily retardation. Now, according to the longitude of the observer, a proportional part of the daily retardation must be allowed to the tabulated meridian passage: thus a person in 60°, or 4h. W. longitude, must allow one-sixth of the daily retardation later than it would be at Greenwich, in addition to his allowance for the longitude in time, and so on; but if the observer be to the eastward of Greenwich by the same number of degrees, he must allow the one-sixth of the retardation in addition to the allowance for longitude in time earlier than the meridian passage at Greenwich, and as the transit takes place between the one on the day in question and that on the preceding day, the retardation between these two days must be found. Hence in both cases, whether the observer be in east or in west longitude, *the difference between the two times of transit at Greenwich from the "Nautical Almanac" between which the one at place happens, must be used to find the daily retardation.*

To find the Greenwich date of the moon's meridian passage.

RULE (*a*) Take the moon's meridian passage from page IV, "Nautical Almanac," for the given astronomical day.

(*b*) *If the longitude be east,* find the difference between this time and that of the meridian passage for the day before. *If the longitude be west,* find the difference between it and that for the succeeding day.

(*c*) Multiply this difference by the number of degrees, &c., in the longitude of the observer and divide the product by 360, this will give the correction for retardation.

(*d*) Reduce the longitude to time.

(*e*) If the longitude be west add (*c*) and (*d*), but if the longitude be east subtract (*c*) and (*d*).

Ex. 170. 1887, December 12, a.m. at ship; find the Greenwich date of the moon's meridian passage in longitude 95° 15′ E.

Meridian passage, Dec. 10	=	20h.	47·5m.
„ „ „ 11	=	21	42·3
Retardation in 360°	=		54·8

Meridian passage.

December 11 =	21h.	42m.	18s.
Retardation	—	14	30
Time transit at place	21	27	48
Long. in time, E. =	− 6	21	0
G. M. T. transit, Dec. 11	15	6	48

Retardation.

Daily 54·8m.
Long. 95·25

6)5219·7

60)869·95

14m. 30s.

Hour Angle of the Moon.—This is obtained from the same formula as the hour angle of a star, viz. :—

$$h = \text{R. A. mean sun} + \text{mean time place} - \text{R. A. moon.}$$

Ex. 171. 1887, July 8th, at 6h. 27m. 43s. a.m. mean time at a place in long. 132° 17′ E. ; find the moon's hour angle.

For Greenwich time.

M. T. at place, July 7		18h.	27m.	43s.
Long. in time, E.	−	8	49	8
Greenwich M. T., July	7	9	38	35

Long. in time.

132° 17′ E.
4

60)529 8

8h. 49m. 8s.

Moon's R. A.

July 7, 9h. =	21h.	10m.	54·75s.
Correction = +		1	22·13
True R. A. =	21	12	16·88

Variation.

In 10m. = + 21·288s.
38m. 35s. = 3·858

60)82·129104

Correction + 1 22·13

R. A. of mean sun.

Sidereal time, July 7 =	7h.	0m.	32·79s.
Acceleration for 9h.		1	28·71
Acceleration for 38m.			6·24
Acceleration for 35s.			·10
R. A. mean sun	7	2	7·84

Hour angle.

R. A. mean sun	7h.	2m.	7·84s.
M. T. at place	18	27	43
R. A. meridian	25	29	50·84
R. A. of moon	21	12	16·88
Hour angle	4	17	33·96

Western hour angle = 64° 23′ 29·4″

Exercise VI.

Ex. 172. Define equation of time, mean solar time, and ap-

parent solar time. Draw a diagram showing the apparent time 9h. a.m., and the mean sun, the equation of time being subtractive. *E.* 1874.

Ex. 173. Given sidereal time 8h. 31m. 10s. and the R. A. of the mean sun 2h. 20m. 30s., what is the mean time? Construct a figure. *E.* 1871.

Ex. 174. Given the hour angle of a star, its R. A. and the R. A. of the mean sun, show by means of a figure how mean time may be found. *Royal Naval College,* 1864.

Ex. 175. Explain the meaning of the column headed "*Sidereal time,*" in the "Nautical Almanac." By what other name may it be called? *Royal Naval College,* 1866.

Ex. 176. Having given the mean time at a known place, show how to find what bright stars in the "Nautical Almanac" will pass the meridian next after that time; illustrate by a diagram. At what time will *Vega* pass the meridian of a place in longitude 120° 30′ E., on June 30th, 1887? *E.* 1869.

Ex. 177. At what time will *Markab* cross the meridian of a place in latitude 47° 35′ N., longitude 120° W., on September 20th, 1887, and at what distance north or south of the zenith? *E.* 1868.

Ex. 178. At what time will *α Leonis* pass the meridian of a place in latitude 0°, and longitude 45° 10′ E., on February 10th, 1887, and at what distance north or south of the zenith? *E.* 1875.

Ex. 179. At what time will *Aldebaran* pass the meridian of a place in longitude 150° E., on October 20th, 1887? What other stars of the first magnitude will pass the same meridian during the next hour? *A.* 1878.

Ex. 1880. At what time on May 12th, 1887, will *Arcturus* pass the meridian of a place in latitude 35° 30′ N., and longitude 130° W., and at what distance north or south of the zenith? *E.* 1876.

Ex. 181. Find at what time *α Canis Majoris* will pass the meridian at Greenwich, the star's R. A. being 6h. 38m. 52s., and the R. A. of the mean sun at Greenwich at mean noon 11h. 6m. 2s. Construct the figure to show the positions of *the first point of Aries,* and of the mean sun in respect to the meridian. *Royal Naval College,* 1863.

Ex. 182. What stars will pass the meridian of a place in longitude 52° E., between the hours of 5 and 8 p.m., on 23rd November, 1887? *E.* 1872.

Ex. 183. What bright stars will pass the meridian of a place in longitude 100° W., on December 2nd, 1887, between the hours of 7 and 9 p.m? *E.* 1873.

Ex. 184. What bright stars will pass the meridian of a place in longitude 35° W., between the hours of 8 and 10 p.m., on July 10th, 1887? *E.* 1874.

Ex. 185. How do you find by help of the "Nautical Almanac" the time of the meridian passage of the moon at a given place (1) when it takes place before, (2) when after midnight? What is the time of the meridian passage of the moon in longitude 120° E., on June 15th, 1887? *E.* 1869.

Ex. 186. Given sidereal time = 11h. 32m. 10s., and the R. A. of the mean sun at mean noon at the place = 0h. 42m. 14·5s.; required the correct mean time.

Royal Naval College, 1864.

Ex. 187. The hour angle of a heavenly body being known, show how mean time at place may be found,—

(1) When the heavenly body is the sun.
(2) When the heavenly body is the moon.

Royal Naval College, 1864.

Ex. 188. What is the R. A. of the star that is on the meridian at 9h. 30m. p.m., the R. A. of the mean sun being 20h. 30m.? and construct the figure. *Royal Naval College*, 1865.

Ex. 189. Explain the rule for finding the time when a heavenly body passes the meridian. *Ex.* At what time will a star whose R. A. is 10h. 30m. pass the meridian, the R. A. of the mean sun being 4h. 30m.? and construct the figure.

Royal Naval College, 1865.

Ex. 190. Given mean time=10h. 14m. 15s. p.m., and the equation of time = 2m. 30s. additive to mean time; required the apparent time. Construct a figure and show the position of the true and mean sun with respect to the meridian; the sun's declination deing supposed to be 10° N.

Royal Naval College, 1866.

Ex. 191. The hour angle of a star = 19h. 7m. 10s., R. A. of the star 6h. 50m. 9s., declination 30° N., R. A. of mean sun at preceding Greenwich mean noon 12h. 4m. 29s. Draw a figure on the plane of the equator and find the exact ship mean time, the longitude being 60° E. *For Lieutenant*, 1873.

Ex. 192. Explain the construction of the table which gives the "Correction in finding moon's meridian passage." Find, without the aid of this table, the Greenwich mean time of the

moon's passage across the meridian of a place in longitude 173° 20′ E., on May 12th, given from the "Nautical Almanac."

Meridian passage, May 11th = 11h. 33m.
" " May 12th = 12h. 23m.

For Lieutenant, 1873.

Ex. 193. Having given sidereal time = 3h. 40m. 20s., and mean time of transit of the first point of Aries (as given in the "Nautical Almanac" for the day) = 14h. 57m. 10·8s.; find the ship mean time at the instant, longitude being 20° W.

For Lieutenant, 1873.

Ex. 194. Having given sidereal time at any instant, show how to find mean time at the same instant. *Ex.* In longitude 54° E. sidereal time was 7h. 53m. 0s.; R. A. of the mean sun at preceding Greenwich mean noon was 14h. 27m. 29s.; find the mean time at the instant, drawing a figure on the plane of the equator. *For Lieutenant*, 1873.

Ex. 195. What is meant by "♈"? Find the exact time of transit of ♈ over the meridian of a place in longitude 70° E. on August 18th, having given the R. A. of the mean sun at Greenwich mean noon on August 18th = 9h. 55m. 46·5s.

For Lieutenant, 1874.

Ex. 196. At what time on October 30th will the star *Regulus* be on the six o'clock hour circle at Greenwich, having given R. A. of the mean sun at Greenwich mean noon October 30th 14h. 35m. 28·9s., R. A. of star 10h. 1m. 36·4s?

For Lieutenant, 1873.

Ex. 197. At what time will the upper meridian passage of the moon take place in latitude 27° 10′ S., longitude 73° 29′ E., on May 27th, 1887, and at what distance from the zenith?

Ex. 198. At what true altitude, in what direction, and at what time will the transit of the moon occur at her upper passage in latitude 7° 29′ N., longitude 132° 10′ W., on July 18th, 1887?

CHAPTER VIII.

On finding latitude—The altitude of the pole is equal to the latitude of the observer—Proof—Latitude and declination found by the circummeridional altitudes of a star—Latitude by the meridian altitude of the sun, a star, a planet, and the moon—Examples—To find the time when the moon and planets are at their maximum altitudes—Effects of this on latitude found by moon and planet—Exercise—Examination.

On finding Latitude.

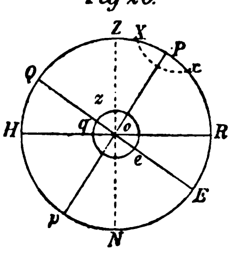
We begin the application of Nautical Astronomy with the methods for determining latitude, because it is the co-ordinate easiest to be found, and also because it is the one used in finding the others

The latitude of a place is the angle at the earth's centre subtended by the arc of a meridian intercepted between the equator and the place. Now as the arc of a declination circle intercepted between the observer's zenith and the equinoctial subtends the same angle, the determination of latitude, in many instances, resolves itself into finding the declination of the observer's zenith. The simplest method of obtaining latitude by observation is by means of meridian altitudes of celestial objects. This we now proceed to illustrate.

The altitude of the pole is equal to the latitude of the observer.

Let $Z\,E\,N\,Q$ be the projection of the heavens on the meridian of the observer; Z the zenith, N the nadir, $E\,Q$ the equinoctial, O the east and west points of the horizon, z the position of the observer on the surface of the earth $z\,e\,q$, $e\,q$ the equator; then $P\,O\,p$ is the axis of the heavens, P the elevated pole, $H\,O\,R$ the plane of the horizon, $z\,O\,q$ the latitude of the observer, which is manifestly equal to $Z\,O\,Q$, the declination of the observer's zenith, and $R\,O\,P$ is the altitude of the pole.

Fig 26.



From the figure :—$R\,O\,P = R\,O\,Z - P\,O\,Z$
$= P\,O\,Q - P\,O\,Z$, because $R\,O\,Z$ and $P\,O\,Q$ are right angles.
$= Z\,O\,Q$.

$\therefore$ Altitude of the pole is equal to the latitude of the observer.

Hence, if by any means the altitude of the elevated pole can be determined, the latitude is at once known. One method is by taking altitudes of a circumpolar star, that is of one whose polar distance is less than the latitude of the observer, and therefore which never sets. One altitude must be taken when the star is on the meridian above the pole, and the other altitude twelve sidereal hours after when the star is on the meridian below the pole. Let X and x be the positions of such a star, then $P\,X$ and $P\,x$ is the polar distance of the star.

$R\,X$ is its altitude at the transit above the pole,
$=$ alt. of the pole $+$ polar distance.

$R\,x$ is its altitude at the transit below the pole,
$=$ alt. of the pole $-$ polar distance.

Adding these two values

$R\,X + R\,x =$ twice the alt. of the pole,
$=$ twice the latitude.

$\therefore$ Latitude $= \frac{1}{2}$ {alt. of star above pole $+$ alt. of star below pole} . . . . I.

The declination of the object may also be found by the same observation, thus :—

Subtracting $R\,x$ from $R\,X$:—

$R\,X - R\,x =$ twice the polar distance.

$\therefore$ Polar dist. $= \frac{1}{2}$ {alt. of star above pole $-$ alt. of star below pole} . . . . II.

This result subtracted from 90° gives the declination.

Ex. 199. If the corrected altitude of β Ursæ Minoris at its upper transit at the Navigation School, Plymouth, be 65° 45′ 23″, and at its lower transit 34° 59′ 27″, find the latitude of the school and the declination of the star.

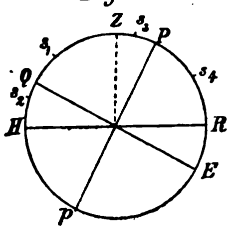
Alt. above the pole	65° 45′ 23″	. . . . . .	65° 45′ 23″
Alt. below the pole	34 59 27	. . . . . .	34 59 27
	2)100 44 50		2)30 45 56
Latitude =	50 22 25	Polar dist. =	15 22 58
		Declination	74 37 2

By this method no other appliance is required than one for

observing altitudes and a horizon; but as the observations can be made only at stationary observatories, and then only at a great inconvenience, recourse is had to a more convenient method in which the declination of the body is used, viz. to the meridian altitude of objects generally.

LATITUDE BY MERIDIAN ALTITUDE.—PROOF. In the annexed figure let the letters denote the same as in the last, but P the north pole, and s_1, s_2, s_3, s_4 be the positions of four celestial objects; $Q s_1$, $Q s_2$, $Q s_3$, $E s_4$ will be the declinations of the objects, $P s$, $P s_2$ will be their polar distances.

Fig 27



First:—Let the object be s_1, where the zenith and declination are both north:—

Then $Q Z$ or latitude $= Z s_1 + Q s_1$,
$= $ zen. dist. $+$ dec. of object . I.

Secondly:—Let the object be s_2, where the zenith is north and declination south:—

Then $Q Z$ or latitude $= Z s_2 - Q s_2$,
$= $ zen. dist. $-$ dec. of object . II.

Thirdly:—Let the object be s_3, where the object is between the pole and the zenith:—

Then $Q Z$ or latitude $= R P$
$= R s_3 - P s_3$
$= $ alt. of object $-$ its polar dist. III.

Fourthly:—Let the object be below the pole, as s_4:—

Then $Q Z$ or latitude $= R P$
$= R s_4 + P s_4$
$= $ alt. of object $+$ its polar dist. IV.

From I. and II. we deduce that when an object is on the meridian, the latitude of the observer is equal to the *sum* of the zenith distance of the object and its declination when the zenith distance and declination are of the same name; but is equal to the *difference* between the zenith distance of the object and its declination when the zenith distance and declination have different names.

From III. we deduce that when the object is between the zenith and the pole the latitude is equal to its altitude minus its polar distance.

From IV. we deduce that when the object is on the meridian

under the pole the latitude is equal to its altitude plus its polar distance.

To find the latitude by the meridian altitude of the sun above the pole.

RULE (1) Find the Greenwich date, and correct the declination.

(2) Correct the altitude and subtract it from 90° to obtain the zenith distance, which mark of an opposite name to the bearing of the sun.

(3) If the zenith distance and declination be of the same name, the sum is the latitude, of the same name as the declination; but if they be of different names, their difference is the latitude, and of the same name as the greater of the two.

To find the latitude by the meridian altitude of the sun below the pole.

RULE (1) Because the sun is under the pole it must be midnight of the given day, hence the time at ship will be twelve hours of that day. To that time apply the longitude in time; the result is the Greenwich time.

(2) Correct the declination for the Greenwich date, and subtract the result from 90° for the polar distance.

(3) Correct the altitude, and add to the polar distance; the latitude is of the same name as the declination.

Ex. 200. 1887, May 29th, in longitude 165° 0′ 20″ E., the observed meridian altitude of the sun's L. L. was 77° 2′ 15″, bearing north. Index error of the sextant − 1′ 15″, height of eye 20 feet. Required the latitude of the place.

For Greenwich time.

App. time place, May	29d.	0h.	0m.	0s.
Long. in time, E. −		11	0	1
App. time Greenwich, May	28	12	59	59

Long. in time.

165° 0′ 20″ E.
4
660 1 20
11h. 0m. 1s.

Sun's declination.

May 29 =	21°	36′	52·6″	N.
Correct. for 11h. −		4	16·4	
True declination	21	32	36·2	N.

Variation of declination.

In 1 hour	23·31″
No. hrs. from noon	10
	60)256·41
	4′ 16·4″

Observed altitude ☉̲	77°	2′	15″	N.
Index error	−	1	15	
	77	1	0	
Dip (20 feet)	−	4	24	
	76	56	36	
Semidiameter	+	15	48·6	
Apparent altitude	77	12	24·6	
Refraction	−		12	
	77	12	12·6	
Parallax		+	2	
True altitude	77	12	14·6	N.
Zenith distance	12	47	45·4	S.
True declination	21	32	36·2	N.
Latitude	8	44	50·8	N.

Ex. 201. On July 2nd, 1887, the observed meridian altitude by artificial horizon on the meridian below the pole of the sun's U. L. in longitude 17° 29′ 30″ W., was 20° 30′ 10″. Height of eye 16 feet, index error − 4′ 16″. What was the latitude?

For Greenwich time.

App. time place, July	2d.	12h.	0m.	0s.
Long. in time, W.	+	1	9	58
App. time Greenwich, July	2	13	9	58

Long. in time.

17° 29′ 30″ W.
4
69 58 0
1h. 9m. 58s.

For sun's declination.

July 3 =	22°	58′	49·1″	N.
Correct. for 10h. 50m.	+ 2		9·9	
True declination	23	0	59	N.

Variation of declination.

In 1 hr. = + 11·99″
10h. 50m. = 10·83
60)129·8517
+ 2 9·9

Observed altitude ☋	20°	30′	10″	N.
Index error	—	4	16	
	2)20	25	54	
	10	12	57	
Semidiameter	—	15	46	
Apparent altitude	9	57	11	
Refraction	—	5	9	
	9	52	2	
Parallax		+	9	
True altitude	9	52	11	N.
Polar distance	66	59	1	N.
Latitude	76	51	12	N.

To find the latitude by the meridian altitude of a star.
Latitude by the meridian altitude of the sun can, except in very high latitudes, be found only once every day; but as the number of stars suitable for observation are very numerous, it follows that latitude by their means may be found as often as one crosses the meridian and the weather is favourable. Owing to the obscurity of the horizon at night, taking the altitudes of stars is always a tedious process; still a good observer can well perform this, especially in bright moonlight or with an artificial horizon; and as the declination of stars change so very slowly, a Greenwich date is unnecessary.

RULE (1) Correct the altitude of the star, and find the zenith distance.

(2) Take the declination from the "Nautical Almanac," and, without correcting it, proceed as in finding latitude by the sun.

Ex. 202. 1887, March 25th, the observed meridian altitude of Canopus (α Argus) in longitude 135° 22′ 15″ W., was 53° 27′ 10″, bearing south. Height of the eye 21 feet, index error + 3′ 17″. Find the ship's position.

Observed latitude	53° 27′ 10″ S.
Index error	+ 3 17
	53 30 27
Dip (21 feet)	− 4 31
Apparent altitude	53 25 56
Refraction	− 42
True altitude	53 25 14 S.
	90
Zenith distance	36 34 46 N.
Declination	52 38 30 S.
Latitude	16 3 44 S.

Ship's position. Latitude 16° 3′ 44″ S.
longitude 135° 22′ 15″ W.

To find the latitude by the meridian altitude of the moon and planets.

The method of determining latitude by the meridian altitudes of the moon and planets is in theory the same as that by the sun and stars; but more difficulties are presented in practice from the fact that, owing to the nearness of the former to our earth, their declinations, parallaxes, and semidiameters change so quickly, that a greater number of corrections are necessary, as has been already explained in the article "On the correction of altitudes." In addition to involving more labour, the method of finding latitude by the moon and planets is less trustworthy than that by the sun and stars. This is owing to the rapid change in their declination, and thus the moon and planets are seldom on the meridian at their maximum altitudes. The rule to be observed is as follows:—

Rule (*a*) Find the Greenwich date of the moon's meridian passage at the given place (p. 98). If the moon's meridian passage under the pole be used for the purpose of finding latitude, the lower meridian passage from page 4, "Nautical Almanac," must be used and corrected in a similar manner to the time of the upper meridian passage.

(*b*) Take from the "Nautical Almanac" the moon's or planet's semidiameter, horizontal parallax, and declination, and correct them for the Greenwich time.

(*c*) Correct the observed altitude and find the zenith distance.

(*d*) *For the upper transit* the latitude is the sum of the zenith distance and declination, if these be of the same name; but is the difference of these quantities if one be north and the other south, and is of the same name as the greater.

For the lower transit. The latitude is the sum of the altitude and the polar distance, and is of the same name as the declination.

Ex. 203. 1887, June 8th, in longitude 68° 20′ E., the observed meridian altitude of the moon's upper limb at the upper transit, bearing S., was 70° 13′ 15″. Index correction + 29″, height of the eye 24 feet. Required the latitude.

Here June 8th is civil time, and as the upper transit took place at 1h. 55·5m. in the morning of that day we get

	Meridian passage on 8th =	7d. 13h. 55·5m.
Long. E.	Meridian passage on 7th =	6 13 0·1
	Retardation in 360°	55·4

For Greenwich date.

	d.	h.	m.	s.
Meridian passage, June	7d.	13h.	55m.	30s.
Retardation		–	10	31
Time transit at place		13	44	59
Long. in time, E.		– 4	33	20
Greenwich time, June	7	9	11	39

Retardation.

For 360° = – 55·4m.
68° 20′ = 68⅓
6)3785·66
60)630·94
– 10m. 31s.

Moon's semidiameter.

At noon, June 7 =	15′ 45·1″
Correction	— 4·7
	15 40·4
Augmentation	+ 14·8
Correct semidiameter	15 55·2

Correction.

For 12 hrs. = — 6·1″
9h. 11m. 39s. 9·2
12)5612
– 4·7

For correcting the horizontal parallax a rough latitude is required. This is found by subtracting the altitude from 90°

for a zenith distance and applying the uncorrected declination. In this case the zenith distance is about 20° N., and declination about $19\frac{3}{4}$° S., and therefore the approximate latitude is about 15′ N.

Moon's horizontal parallax.

At noon, June 7	=	57′ 42·5″
Correction		− 17·1
		57 25·4
Reduction (lat. 0)		= 0
Correct horizontal parallax		57 25·4 = 3445·4″

Correction.

For 12 hrs.	− 22·3″
9h. 11m. 39s.	9·2
	12)205·16
	− 17·1

Moon's declination.

June 7d. 9h.	=	19° 44′ 39·7″ S.
Correct. for 39m. 11s.	=	— 20·3
Correct declination	=	12 44 11·4 S.

Variation.

For 10m. =	− 5·19
39m. 11s.	3·92
	− 20·3448

Observed alt. U. L.	= 70° 13′ 15″ S.		
Index error	+ 29		
	70 13 44		
Dip (24 feet)	− 4 49		
	70 8 55		
Semidiameter	− 15 55·2		
Apparent altitude	69 52 59·8	. . . . .	Cos. 9·536474
Refraction	− 21	Hor. parallax = 3445·4	log. 3·537239
	69 52 39·8		
Parallax in alt.	+ 19 45·0 =	1185·0	log. 3·073713
True altitude	70 12 23·8 S.		
	90 0 0		
Zenith distance	19 47 36·2 N.		
True declination	19 44 11·4 S.		
Latitude	0 3 24·8 N.		

Ex. 204. 1887, December 16th, at 8h. 47m. 15s. a.m. mean time at ship, in longitude 27° 30′ W., the observed meridian altitude of the L. L. of Venus was 28° 27′ 45″, zenith north of the planet. Height of eye 14 feet, index correction − 2′ 51″. Find the latitude.

December 15th, Meridian passage of Venus			20h.	47m.	12s.
„ 16th, „ „ „ „			20	47	30
Retardation for 360°	=			+	18
∴ Retardation for 27° 30′ W.	=			+	1·4

For Greenwich time.

Mer. pas. Dec.	15d.	20h.	47m.	12s.
Retardation			+	1·4
Mer. pas. at place		20	47	13·4
Long. 27° 30′ W.		1	50	0
Greenw. M. time	15	22	37	13·4

For Venus' declination.

Dec. noon, 15 Dec.	11°	19′	1·3″ S.
„ „ 16 „	11	39	4·8 S.
Diff. dec. in 24h.	+ 20	3·5	
∴ Diff. dec. in 22·62h.	+ 18	54·3	
Dec. noon, 15th	11	19	1·3 S.
True dec.	11	37	55·6 S.

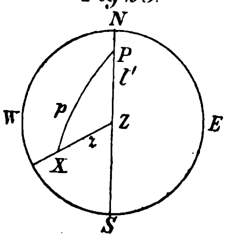
Observed altitude	28°	27′	45″ S.
Index error	−	2	51
	28	24	54
Dip (14 feet)	−	3	41
	28	21	13
Semidiameter		+	10·5
Apparent altitude	28	21	23·5
Refraction	−	1	45·6
	28	19	37·9
Parallax in alt.		+	9·9
True altitude	28	19	47·8 S.
Zenith distance	61	40	12·2 N.
True declination	11	37	55·6 S.
Latitude	50	2	16·6 N.

Data Nautical Almanac.

Semidiameter 10·5″ }
Hor. parallax 11·2 } p. 277.

. . . . .		Cos. 9·944487
Hor. parallax 11·2″	log.	1·049218
Par. in alt. 9·86	log.	0·993705

Fig 28.



To find the time when the moon and planets are at their maximum altitudes.

Let $N\,E\,S\,W$ be the projection of the celestial concave on the plane of the horizon, P the elevated pole, Z the zenith of the observer, h the hour angle of the object X, p its polar distance, z its zenith distance, and l' the colatitude. Then :—

$$\text{Cos. } z = \cos. p. \cos. l' + \sin. p. \sin. l'. \cos. h \quad . \quad . \quad \text{I.}$$

Let p' and z' be the small changes in the polar and zenith

distances corresponding to the small change h' in the hour angle; then

$$\begin{aligned}\text{Cos.}\,(z + z') &= \cos.\,(p + p') \,.\, \cos.\, l' \\ &\quad + \sin.\,(p + p') \,.\, \sin.\, l' \,.\, \cos.\,(h + h'),\end{aligned}$$

$$\begin{aligned}\text{Cos.}\, z \,.\, \cos.\, z' - \sin.\, z \,.\, \sin.\, z' &= (\cos.\, p \,.\, \cos.\, p' - \sin.\, p \,.\, \sin.\, p') \cos.\, l' \\ &\quad + (\sin.\, p \,.\, \cos.\, p' + \cos.\, p \,.\, \sin.\, p') \\ &\quad \times (\cos.\, h \,.\, \cos.\, h' - \sin.\, h\, \sin.\, h') \sin.\, l'.\end{aligned}$$

Now as p', z' and h' are very small quantities, their cosines may be taken respectively equal to unity, then

$$\begin{aligned}\text{Cos.}\, z - \sin.\, z \,.\, \sin.\, z' &= (\cos.\, p - \sin.\, p \,.\, \sin.\, p') \cos.\, l' \\ &\quad + (\sin.\, p + \cos.\, p \,.\, \sin.\, p') (\cos.\, h - \sin.\, h \,.\, \sin.\, h') \sin.\, l' \\ &= (\cos.\, p - \sin.\, p \,.\, \sin.\, p') \cos.\, l' \\ &\quad + (\sin.\, p \,.\, \cos.\, h - \sin.\, p \,.\, \sin.\, h\, \sin.\, h' \\ &\quad + \cos.\, p \,.\, \sin.\, p' \cos.\, h - \cos.\, p \,.\, \sin.\, p' \,.\, \sin.\, h \,.\, \sin.\, h') \\ &\qquad \times \sin.\, l' \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{II.}\end{aligned}$$

Subtracting II. from I.—

$$\begin{aligned}\text{Sin.}\, z \,.\, \sin.\, z' &= \sin.\, p \,.\, \sin.\, p' \cos.\, l' + (\sin.\, p \,.\, \sin.\, h \,.\, \sin.\, h' - \\ &\quad \cos.\, p \,.\, \sin.\, p' \cos.\, h + \cos.\, p \,.\, \sin.\, p' \,.\, \sin.\, h \,.\, \sin.\, h') \sin.\, l',\end{aligned}$$

but as $\cos.\, p$ and $\sin.\, h$ are small fractions, and $\sin p'$ and $\sin h'$ are each almost equal to zero, the omission of the term in which these occur as factors may be made without introducing any material error; then

$$\begin{aligned}\sin.\, z \,.\, \sin.\, z' &= \sin.\, p \,.\, \sin.\, p' \cos.\, l' + \sin.\, p \,.\, \sin.\, h \,.\, \sin.\, h' \sin.\, l' \\ &\quad - \cos.\, p \,.\, \sin.\, p' \cos.\, h \,.\, \sin.\, l'.\end{aligned}$$

Now the altitude of any object will be greatest when its zenith distance is least, that is, when $\sin.\, z$ is zero, or when the left-hand side of the above equation vanishes; then

$$\begin{aligned}&\sin.\, p\, \sin.\, h \,.\, \sin.\, h' \sin.\, l' \\ &\quad = -(\sin.\, p \,.\, \cos.\, l' - \cos.\, p \,.\, \cos.\, h \,.\, \sin.\, l') \sin.\, p'.\end{aligned}$$

At the maximum altitude the body is so near to the meridian, or h is so small, we may take its cosine as unity, then

$$h'' \,.\, \sin.\, 1'' = -\frac{\sin.\,(p - l')}{\sin.\, p \,.\, \sin.\, l'} \cdot \frac{p' \,.\, \sin.\, 1''}{h' \,.\, \sin.\, 1''}$$

$$\therefore\; h'' = -\frac{\sin.\,(p - l')}{\sin.\, 1'' \,.\, \sin.\, p \,.\, \sin.\, l'} \cdot \frac{p'}{h'};$$

$$\text{i.e. } h \text{ (sec. of time)} = -\frac{1}{15 \,.\, \sin.\, 1''} \cdot \frac{\sin.\,(p - l')}{\sin.\, p \,.\, \sin.\, l'} \cdot \frac{p'}{h'}.$$

To those familiar with the differential calculus, the solution of the problem is much easier, and is as follows:—

Beginning with the fundamental formula

$$\cos. z = \cos. p \,.\, \cos. l' + \sin. p \,.\, \sin. l' \cos. h.$$

Differentiating, and considering l' as the only constant quantity in the equation, because as p varies, both z and h must also vary, and we have to find h when z is a minimum.

$$\therefore -\sin. z.dz = -\sin. p.dp.\cos. l' + \cos. p.dp.\sin. l' \cos. h$$
$$-\sin. p \,.\, \sin. l' \sin. h \,.\, dh;$$

$$\therefore \sin. z \frac{dz}{dh} = (\sin. p \,.\, \cos. l' - \cos. p \,.\, \sin. l' \,.\, \cos. h) \frac{dp}{dh}$$
$$+ \sin. p \,.\, \sin. l' \,.\, \sin. h.$$

This $= o$ because z is to be a minimum.

$$\text{Hence } \sin. h = -\frac{\sin. p.\cos. l' - \cos. p.\sin. l'.\cos. h}{\sin. p \,.\, \sin. l'} \cdot \frac{dp}{dh} \quad . \text{ I.,}$$

where dp is the change of polar distance corresponding to the change of hour angle dh; thus dh being expressed in arc, if it represents a second in space, then $\frac{dp}{dh}$ is the change in polar distance in one-fifteenth of a second of time.

At the maximum altitude of any body it is so near the meridian, or h is so small we may take its cosine equal to unity and $\sin. h = h''. \sin. 1''$, then we have from I.—

$$h''. \sin. 1'' = -\frac{\sin. p \,.\, \cos. l' - \cos. p \,.\, \sin. l'}{\sin. p \,.\, \sin. l'} \cdot \frac{dp}{dh} \text{ in arc;}$$

$$\therefore h = -\frac{1}{15 \,.\, \sin. 1''} \cdot \frac{\sin. (p - l')}{\sin. p \,.\, \sin. l'} \cdot \frac{dp}{dh} \text{ in time,}$$

which is the interval that should be applied to the time of the meridian passage to give the time of maximum altitude.

Ex. 205. 1887, October 14th. Find the time when the moon will have the greatest altitude at the Navigation School, Plymouth, in latitude 50° 22′ 25″ N., longitude 4° 7′ 16″ W.

For meridian passage.

October	14d.	22h.	32m.	6s.
Retardation				+ 38
		22	32	44
Longitude, W.		+	16	29
G. M. time, October	14	22	49	13

For retardation.

For 360° = 55·4m.

Long.	4·12
	1108
	554
	2216
60)	228·248
6)	3·804
	·634 = 38s.

For moon's declination.

October 14d. 22h. =	3° 11′ 10·6″ N.
Correction	− 10 30·7
True declination	3 0 39·9 N.

Correction for declination.

For 10m. =	− 128·19″
49m. 12s. =	4·92
	25638
	115371
	51276
	60)630·6948
	10′ 30·7″

∴ N.P.D. =	86° 59′ 20″
l' or colat. =	39 37 25
∴ $p - l'$ =	47 21 45

Very approximately—

$$\frac{dp}{dh} = \frac{\text{change in declination in 10m.}}{15 \times \text{no. seconds in 10m.}} = \frac{128 \cdot 19}{15 \times 10 \times 60} = \cdot 01424\dot{3},$$

$$\text{and } h \text{ seconds} = \frac{1}{15 \,.\, \sin. 1''} \cdot \frac{\sin. (p - l')}{\sin. p \,.\, \sin. l'} \cdot \frac{dp}{dh}$$

$$= \frac{1}{15 \,.\, \sin. 1''} \cdot \frac{\sin. 47^\circ\ 21'\ 45''}{\sin. 86^\circ\ 59'\ 20'',\ \sin. 39^\circ\ 37'\ 35''} \times \cdot 01424\dot{3}.$$

For sin. $(p - l')\dfrac{dp}{dh}$.

$p - l' = 47° 21′ 45″$	sin.	9·866673
$\frac{dp}{dh} = \cdot 01424\dot{3}$	log.	$\bar{2}$·153510
sin.$(p-l')\frac{dp}{dh}$	log.	8·020183
15 sin.1″sin.p . sin.l'	log.	5·665736
h seconds 226·2	log.	2·354447
		= − 3m. 46·2s.

For 15 . *sin.* 1″ *sin. p . sin. l′.*

15	log.	1·176091
1″ sin.	log.	4·685575
p = 86° 59′ 20″	sin.	9·999400
l' = 39 37 35	sin.	9·804670
15 sin. 1″ sin. p . sin. l'	log.	5·665736

Now because p is increasing $\frac{dp}{dh}$ is positive, and hence the minus sign which the formula gives must be placed before the result found. This shows that the maximum altitude occurred 3m. 46·2s. *before* the time of the meridian passage. In this time the declination will have decreased 128·19″ × ·377 = 48·3″, and therefore, if the maximum altitude be considered the meridian altitude, it will be too great by 48·3″, and hence the latitude deduced from that altitude will be too small by that amount.

EXERCISE VII.

Ex. 206. 1887, February 14th, in longitude 104° 28′ 15″ W. the observed meridian altitude of the sun's lower limb was

79° 26′, bearing south; the index error of the sextant −43″; height of the eye above the sea 21 feet. Required the latitude.

Ex. 207. 1887, September 23rd. The observed meridian altitude of the sun's lower limb was 50° 2′ 10″ in longitude 46° 11′ E. zenith south of the sun; height of eye 25 feet; index correction − 3′ 28″. Find the latitude.

Ex. 208. 1887, June 21st, in longitude 13° 25′ E. by artificial horizon the altitude of the sun's upper limb when under the pole was 17° 25′ 10″; height of eye 19 feet; index error + 2′ 53″. Required the latitude.

Ex. 209. 1887, July 27th, in longitude 150° 20′ W. the observed meridian altitude of the lower limb of the sun was 73° 0′ 15″, the zenith being south of the sun; index correction −3′ 17″; height of the eye 27 feet. What is the latitude?

Ex. 210. 1887, April 6th, at about 8h. 16m. p.m., in longitude 117° 20″, the observed meridian altitude of the star ι Argus, bearing south, was 58° 47′ 20″; height of the eye 23 feet; index error of the sextant −37″. Required the latitude.

Ex. 211. 1887, August 5th. The observed meridian altitude of γ Ursæ Majoris under the North Pole was 19° 18′ 50″; index correction + 1′ 10″; height of eye above the sea 21 feet; Required the latitude. *For Junior Officers Afloat,* 1881.

Ex. 212. 1887, December 26th, at about 6h. 21m. a.m. mean time at ship in longitude 135° 0′ 15″ E., the observed meridian altitude of the lower limb of the planet Mars was 50° 58′ 45″ zenith north of the planet; index error + 1′ 47″; height of the eye 26 feet. Find the latitude.

Ex. 213. 1887, November 23rd. The observed meridian altitude of the moon's L. L. was 61° 5′, zenith south of the moon; longitude of the ship 45° 14′ E.; index correction for the sextant − 3′ 29″; height of eye 17 feet. Required the latitude.

Ex. 214. 1887, January 5th, in longitude 30° 10′ W., the observed altitude of the moon's lower limb was 32° 57′ 20″, when on the meridian, bearing N.; height of the eye above the sea 22 feet; index error + 2′ 53″. Find the latitude.

Ex. 215. 1887, October 22nd. The observed meridian altitude of the moon's U. L. under the South Pole was 12° 49′ 20″; longitude of the observer 58° 15′ W.; height of the eye 21 feet; index correction for sextant + 5′ 38″. Find the latitude.

Ex. 216. 1887. The observed meridian altitude of the sun's upper limb on March 28th was 69° 57′, bearing south; height of the eye 21 feet; index error + 4′ 55″. Required the latitude if the longitude be 135½° E.

Ex. 217. 1887, April 1st. The observed meridian altitude of α' Crucis under the South Pole, taken with an artificial horizon, was $29° 43' 15''$; index correction for the sextant $+ 2' 35''$; height of eye above the sea 17 feet. Required the latitude.

Ex. 218. 1887. The observed meridian altitude of Jupiter's U. L. on April 27th at about 11h. 29m. p.m., in longitude $50° 20'$ W., was $63° 27' 50''$, bearing north; height of eye 24 feet; index correction $+ 2' 10''$. Find the latitude.

Ex. 219. 1887, September 5th. The observed altitude of the moon's L. L. when on the meridian was $61° 29' 30''$ zenith north of the moon; index error of sextant $+ 5' 28''$; height of eye above the sea 19 feet; longitude of the ship $39° 25'$ W. Required the latitude.

Ex. 220. What are circumpolar stars, and how are they employed to determine the latitude of a place on the earth's surface? *B.A. and B.Sc. London*, 1873.

Ex. 221. A circumpolar star passes the zenith of a place, and its altitude at the inferior transit $= 20°$. Required the latitude. *Royal Naval College*, 1863.

Ex. 222. Find the rule for finding the latitude by a meridian altitude of a heavenly body above the pole, and draw diagrams for the following cases :—(1) latitude of the place N., declination of the body N., body S. of zenith; (2) latitude N., declination N., body N. of zenith; (3) latitude N., declination S.; (4) latitude S., declination S., body N. of zenith; (5) latitude S., declination S., body S. of zenith; (6) latitude S., declination N. *E.* 1871.

Ex. 223. A star being supposed at its upper meridian transit to pass through the zenith, and at its lower to touch the horizon of the place of observation; determine on elementary principles of spherical trigonometry the latitude of the place and the declination of the star. *B.A. and B.Sc. London*, 1878.

Ex. 224. The latitude of a place is equal to the altitude of the pole above the horizon, and also to half the sum of the meridian altitudes of a circumpolar star above and below the pole. Required the proofs. *Royal Naval College*, 1864.

Ex. 225. Prove the rules for finding the latitude of a place by the observation of a meridian altitude of a body above the pole. Give diagrams for the following cases :—(1) when the zenith is between the elevated pole and the body, and the latitude and declination are of the same name; (2) when the zenith is between the elevated pole and the body, and the latitude and declination of different names; (3) when the body is between the pole and the zenith. *E.* 1876.

Ex. 226. If a star passes through the zenith of a place, what relation connects the declination of the star with the latitude of the place? *B.A. and B.Sc. London*, 1877.

Ex. 227. Given the meridian altitude of a heavenly body below the pole $= 42° 42' 10''$, and its declination $64° 42'$ N. Construct a figure and find by calculation the latitude.

Royal Naval College, 1863.

Ex. 228. Explain how to determine latitude at sea by meridian observations of the sun.

The apparent altitude of the sun's lower limb at noon on August 7th is observed to be 30°, the refraction for apparent altitude of 30° is $1' 40''$; the sun's declination at Greenwich is $16° 24' 23''$ at apparent noon on the 7th, and $16° 7' 26''$ on the 8th, and his semidiameter is $15' 49''$, the longitude being about 30° west of Greenwich. What is the latitude?

Final B.Sc. London, 1879.

Ex. 229. Show by a diagram how the latitude and longitude enable you to determine the position of a place. Prove the rule for finding the latitude by the meridian altitude of the sun. Why is the meridian altitude the greatest during the day?

E. 1877.

Ex. 230. In determining the latitude at sea, by means of an observed meridian altitude of the sun, calculate approximately, in nautical miles, the error in latitude which would result from an error of one minute in the altitude, as given by observation.

B.A. and B.Sc. London, 1878.

Ex. 231. Prove that the latitude of a place on the earth's surface is equal to the altitude of the elevated pole at that place. Show how to find the latitude of a place by observing the altitude of the sun above the pole, distinguishing between the cases in which the latitude and declination are of the same or of different names. *E.* 1881.

Ex. 232. In what latitude north will the sun make his inferior transit at a point 18° below the horizon when at extreme north declination? *Royal Naval College*, 1867.

Ex. 233. Explain exactly how far the position of a star in the heavens may be determined, from the single circumstance of its either passing through the zenith, or just reaching without sinking below the horizon of a place of known latitude on the surface of the earth. *Final B.Sc. London*, 1877.

CHAPTER IX.

Latitude deduced from altitudes near the meridian—Reduction to the meridian—Proof of formula—Direct method of exmeridian altitude—Proof of formula—Comparison of the two methods—How the hour angle is found—On the most advantageous time for observations for latitude near the meridian—Rules—Examples—Exercise—Examination.

ON FINDING LATITUDE BY THE ALTITUDE OF AN OBJECT NEAR THE MERIDIAN.

FOR some reason or other, the time for taking the meridian altitude of a celestial object may have passed without latitude having been determined: or, as is often the case, the body may have been perfectly visible some little time before or after its transit, but at that instant may have become totally obscured; then to secure latitude for the day recourse must be had to other means than a meridian altitude.

There are two methods in general use under the term *circum-meridional altitude* for determining the latitude by a single altitude near the meridian, and in importance and simplicity these are second only to the meridian altitude.

(1) *By the reduction to the meridian.* This consists in computing a correction to be applied to the altitude of a body out of the meridian, in order to determine what its altitude would be on the meridian. In this calculation the latitude by account always enters; so that, if the latitude by account be very erroneous, an incorrect result must follow; the work must be repeated, using the latitude found by computation instead of the latitude by account until the required accuracy is obtained. It is seldom necessary that more than one repetition should be used.

(2) *By a direct process.* This consists in the application of rules deduced directly from spherical trigonometry into which latitude by account does not enter.

PROOF FOR COMPUTING THE REDUCTION TO THE MERIDIAN.—The data used are the hour angle of the object, its declination and altitude, and the latitude by account.

In the annexed figures, one on the plane of the meridian and

Fig 29

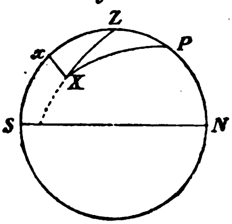
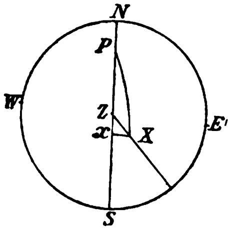


Fig 30.



the other on the plane of the horizon, let P be the pole, Z the zenith, X the position of the object when observed, and x its place when on the meridian:

$$\begin{aligned} \text{Then } PX &= \text{the polar distance} = p, \\ ZX &= \text{the zenith distance} = z, \\ ZPX &= \text{the hour angle} = h, \\ ZP &= \text{the colatitude} = l', \end{aligned}$$

and $l =$ the latitude.

Let r be the reduction to the meridian, that is the difference between the altitude of the object when observed and when on the meridian, or what is the same thing, the difference between the zenith distances at these times: and therefore the meridian zenith distance $= Z - r$.

$$\text{Then} \qquad \text{Cos. } h = \frac{\text{cos. } z - \text{cos. } p \,.\, \text{cos. } l'}{\text{sin. } p \,.\, \text{sin. } l'}$$

$$= \frac{\text{cos. } z - \text{sin. } d \,.\, \text{sin. } l}{\text{cos. } d \,.\, \text{cos. } l} \quad . \quad . \quad . \quad . \quad \text{I.}$$

But when the object is on the meridian the hour angle vanishes, $h = o$ and cos. $h = 1$; and z becomes the meridian zenith distance $z - r$, where r is a very small quantity, because the observation is supposed to be made when the object is near the meridian. Making these substitutions the formula becomes

$$1 = \frac{\text{cos. } (z - r) - \text{sin. } d \,.\, \text{sin. } l}{\text{cos. } d \,.\, \text{cos. } l} \quad . \quad . \quad . \quad \text{II.}$$

Subtract I. from II.—

$$1 - \cos. h = \frac{\cos. (z - r) - \cos. z}{\cos. d . \cos. l}$$

$$2 \sin.^2 \frac{h}{2} = \frac{2 \sin. \left(z - \frac{r}{2}\right) . \sin. \frac{r}{2}}{\cos. d . \cos. l}.$$

Now as r is so small we may without material error write $\sin. (z - r)$ for $\sin. \left(z - \frac{r}{2}\right)$ and $\frac{r''}{2} \sin. 1''$ for $\sin. \frac{r}{2}$, then—

$$\frac{r''}{2} \sin. 1'' = \sin.^2 \frac{h}{2} . \cos. d . \cos. l . \operatorname{cosec.} (z - r);$$

$$\therefore r'' = \sin.^2 \frac{h}{2} . \cos. d . \cos. l . \operatorname{cosec.} (\text{mer. zen. dist.}) . \frac{2}{\sin. 1''}.$$

The changes we have introduced render the formula we have obtained only an approximation, but if taken within proper limits, the error introduced will be less than those incidental to all observations made at sea and is *practically* perfect; but if the latitude thus obtained differs much from the latitude by account, the calculation must be repeated, using the computed latitude for the latitude by account. Another difficulty also presents itself in finding the meridional zenith distance; but this is obviated by calculating it from the latitude by account and the declination.

$$\text{Merid. zen. dist.} = l \pm d,$$

according as the latitude and declination are of different or of the same name. It must also be noticed that $\frac{2}{\sin. 1''}$ is a constant quantity, and may be thus calculated:—

$$\begin{aligned} \log. \frac{2}{\sin. 1''} &= \log. 2 - \log. \sin. 1'' \\ &= \cdot 301030 - \bar{6} \cdot 685575 \\ &= 5 \cdot 615455. \end{aligned}$$

Our formulæ finally becomes—

$$\log. r'' = 2 \log. \sin. \frac{h}{2} + \log. \cos. d + \log. \cos. l$$
$$+ \log. \operatorname{cosec.} \text{mer. zen. dist.} + 5 \cdot 615455 - 40.$$

Proof for the direct Method of computing Latitude from an object near the Meridian.—The data required are, the altitude of the object, the apparent time at ship, and the declination.

Fig 31

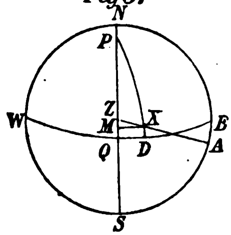
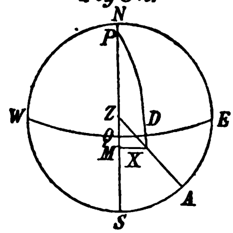


Fig 32.



Let $N E S W$ be a projection of the celestial concave on the plane of the horizon, $N S$ the meridian, $W Q E$ the equinoctial, P the elevated pole, Z the zenith, X the object out of the meridian. Then $A X$ is its altitude, $Z X$ its zenith distance, $D X$ its declination, $P X$ its polar distance, $M P X$ its hour angle. From X draw $X M$ at right angles to $N Z S$.

Then: Latitude of observer $Z Q = Z M \pm M Q$ . . (1)

We shall call $M Q$ the first, and $Z M$ the second arc.

In the right-angled triangle $P M X$ are given the hour angle $M P X = h$, and the polar distance $P X = p$.

and $\cos. M P X = \cot. P X \,.\, \tan. P M$.

$\therefore \cot. P M = \sec. M P X \,.\, \cot. P X$.

or $\tan. M Q = \sec. M P X \tan. D X$.

(I) Hence: $\tan.$ first arc $= \sec. h \,.\, \tan. d$ . . . . . (2)

It is evident, from the figures, that $Q M$, or the first arc, is of the same name as the declination.

In the right-angled triangles $P M X$ and $Z M X$, we have

$$\cos. M X = \frac{\cos. P X}{\cos. P M} \qquad \text{and } \cos. ZM = \frac{\cos. Z X}{\cos. M X}$$

$$= \sec. P M \,.\, \cos. P X \qquad = \sec. M X \,.\, \cos. Z X$$

$$\therefore \sec. M X = \cos. P M \,.\, \sec. P X \qquad = \cos. P M \,.\, \sec. P X \,.\, \cos. Z X$$

$$= \sin. Q M \,.\, \operatorname{cosec}. D X \,.\, \sin. A X.$$

Hence—

(II) Cos. second arc $=$ sin. first arc $\times$ cosec. dec. $\times$ sin. alt. (3), and, like the meridian altitude, is of a contrary name to the bearing of the object.

(III) From (1)—Latitude $=$ second arc $\pm$ first arc, according as they are of the same or different names.

COMPARISON OF METHODS.—In the reduction to the meridian latitude by account is indispensable; but in the direct method

it is not necessary. The former is only an approximation (though a very near one), the latter is perfectly rigid. The former is applicable only when near the meridian, the latter can be used at any time if the data be correct. The former is not applicable when the object transits near the zenith, because the motion of a heavenly body in altitude is then very rapid, the meridian zenith distance is small, and the zenith distance at observation is changing very rapidly: hence $z - r$ may not nearly express the meridian zenith distance when calculated from $l \pm d$ where l is the latitude by account.

How the Hour Angle is found.—When observations are made in the morning for longitude by chronometer, the time by clock or watch should also be noted; then when the apparent time at ship is found in the process of working for longitude, the error of the watch for the ship's apparent time should always be registered, as this involves no additional labour. Then with the ship's run between the time thus found and the time of taking observations for circummeridional altitudes will give the apparent time at ship when the latter observations were made; and the hour angle will be known with the least possibility of error. Another method for finding the hour angle if the longitude be known is to take the times of observation by a chronometer whose error and rate are known. This gives Greenwich mean time of observation, from which, by the application of the equation of time and the longitude, the apparent time at ship can be deduced and thus the hour angle obtained. If the object passes near the zenith of a fixed observatory, half the interval between the times of equal altitudes of the object when east and when west will give the hour angle; but at sea the correction for the differences of latitude and longitude made in the interval must be allowed for, and the method then becomes cumbersome.

To find the time most advantageous for deducing latitude from altitudes of celestial objects near the meridian.

In the formula for the reduction to the meridian

$$r'' = \sin.^2 \frac{h}{2} \,.\, \cos.\, d \,.\, \cos.\, l \;\; \text{cosec. (m. z. d.)} \,.\, \frac{2}{\sin. 1''},$$

it is seen that r varies as $\sin.^2 \frac{h}{2}$, but as h is very small, r varies as h^2 nearly, that is, as the square of the interval between the time of taking the observation and the time of transit. Thus, if the hour angle be doubled, the reduction will be four times as

great nearly, and a small error introduced by taking the wrong time will be greatly increased if the hour angle be large.

But this may also be shown independently, thus:—

The zenith distance must vary with the time from transit; let z be the zenith distance corresponding to the hour angle h, and z' be the zenith distance corresponding to the hour angle h', then $z - z'$ is the error in the zenith distance corresponding to the error $h - h'$ in the hour angle; but as the observations are taken near noon, $z - z'$ will give the error in latitude nearly.

$$\text{Now, } \cos. h = \frac{\cos. z - \cos. p \,.\, \cos. l'}{\sin. p \,.\, \sin. l'},$$

$$\text{and } \cos. h' = \frac{\cos. z' - \cos. p \,.\, \cos. l'}{\sin. p \,.\, \sin. l'};$$

$$\therefore \cos. h - \cos. h' = \frac{\cos. z - \cos. z'}{\sin. p \,.\, \sin. l'},$$

$$\text{and } 2 \sin. \frac{h' - h}{2} . \sin. \frac{h + h'}{2} = \frac{2 \sin. \frac{z' - z}{2} . \sin. \frac{z + z'}{2}}{\sin. p \,.\, \sin. l'}.$$

Now because the difference between h and h', and also between z and z', are small, we may, without material error, write on the different sides of the equation h for $\frac{h + h'}{2}$, and z for $\frac{z + z'}{2}$,

$$\text{then } \sin. h \,.\, \sin. \frac{h - h'}{2} = \frac{\sin. z \,.\, \sin. \frac{z - z'}{2}}{\sin. p \,.\, \sin. l'}.$$

Now if Z be the azimuth of the body at observation,

$$\frac{\sin. h}{\sin. Z} = \frac{\sin. z}{\sin. p},$$

substituting this we get

$$\sin. h \,.\, \sin. \frac{h - h'}{2} = \frac{\sin. h . \sin. \frac{z - z'}{2}}{\sin. Z \,.\, \sin. l'};$$

but $\frac{h - h'}{2}$ and $\frac{z - z'}{2}$ are each so small, we may write

$$\frac{h - h'}{2} \sin. 1'' = \frac{\frac{z - z'}{2} \sin. 1''}{\sin. Z \,.\, \sin. l'},$$

$$\therefore z - z' = (h - h') \sin. Z \,.\, \sin. l',$$

that is,—

Error in lat. = error in hr. ∠ × sin. az. × cos. lat. . (A)

From this it will be seen the error in latitude will be least when the azimuth is so; but this is the case when the object is on the meridian, and the nearer the object is to the meridian the less will the error in latitude be by this method. The formula (A) also shows that the error in latitude varies as the cosine of the latitude, i.e. will decrease as the latitude increases, hence this method of determining latitude from an altitude near the meridian is most useful in high latitudes. As a rule, altitudes for latitude by circummeridional altitudes should be taken within twenty minutes of the body's transit, or when the object's azimuth is not more than one point.

For practical purposes the limit of time within which an altitude should be taken is thus determined. The number of minutes in the hour angle should never exceed the number of degrees in the meridional zenith distance calculated from $l \pm d$.

If latitude be calculated from altitudes of objects when near the meridian, first when east of it and then when west of it, and a mean of the results be taken, this method is susceptible of very great accuracy, because the error in latitude which arises from an error in the hour angle will be almost eliminated, as, what one hour angle is too great on one side of the meridian the other hour angle must be too small on the other side; and the mean of the results will not be far from correct. In practice it is preferable to take several altitudes and times, and in calculating use the mean of these, and it will be found this is of greater value than that of any single observation, even a meridian one.

The methods to be used in practice are as follows:—

Rule for the Reduction to the Meridian.—(1) Find the Greenwich date corresponding to the mean of the times of observation and the hour angle.

(2) Correct the declination, and, if necessary, the equation of time, by the Greenwich date.

(3) If the latitude by account and the declination be of different names, take their sum; but if they be of the same name take their difference for the meridional zenith distance to be used in the solution (m. z. d.).

(4) Take the mean of the observed altitudes, and from it find the true altitude (a).

(5) Compute the reduction to the meridian in seconds by adding together—

Twice log. sin. half the hour angle $\left(\log. \sin.^2 \frac{h}{2}\right)$,
log. cos. declination (log. cos. d),
log. cos. latitude by account (log. cos. l),
log. cosec. mer. zen. dist. (log. cosec. m. z. d.),
and constant log. 5·615455.

(7) To the true altitude add the reduction thus found if the body be above the pole, but subtract it if the body be below the pole, the result is the meridian altitude of the body. The latitude is then found as from objects on the meridian.

Ex. 234. 1887, October 4th, a.m., at ship in latitude by account 55° 10′ S., longitude 140° 30′ E., the observed altitudes of the the sun's lower limb, bearing north, taken near the meridian were as follows:—

Time by watch.	*Observed altitudes.*
9h. 45m. 56s. a.m.	⊙ 38° 29′ 0″
9 47 15	⊙ 38 29 50
9 48 31	⊙ 38 30 40

The index error of the sextant was + 5′ 13″; height of the eye above the sea 21 feet. The watch had been found to be slow 2h. 1m. 20s. of apparent time at ship, and the difference of longitude made to the eastward was 23½ miles, after the error on apparent time had been determined. Required the latitude:—

Time by watch.

		9h. 45m.		56s.
			47	15
			48	31
	3)		141	42
Mean of times	3d.	21	47	14
Watch slow	+	2	1	20
	3	23	48	34
Run 23½ miles		E. +	1	34
App. time ship	3	23	50	8
Hour angle			9	52
In arc			2° 28′ 0″	

Observed altitudes.

⊙	38°	29′	0″
⊙		29	50
⊙		30	40
	3)	89	30
Mean of alts.	38	29	50
Index error	+	5	13
	38	35	3
Dip (21 feet)	−	4	31
	38	31	32
Semidiameter	+	16	2
	38	47	34
Refraction	−	1	11
	38	46	23
Parallax		+	7
True altitude	38	46	30

For Greenwich date.

App. time ship	3d.	23h.	50m.	8s.
Long., E.		− 9	22	0
Greenwich app. time	3	14	28	8

Long. in time.

140° 30′ E.
4

60)562 0

9h. 22m. 0s.

Sun's declination.

October 4	=	4°	19′	37·9″	S.
Correction	= −		9	12·1	
True declination	=	4	10	25·8	S.

Variation of declination.

In 1 hour = — 57·93″
9h. 31m. 52s. = 9·53

60)552·0729

9′ 12·1″

For latitude.

$\frac{h}{2}$ =	1°	14′	0″	sin.	8·332924
					2
					16·665848
dec. =	4	10	26	cos.	9·998847
lat. by acct. =	55	10	0	cos.	9·756782
m. z. d. =	51	0	0	cosec.	·109497
$\frac{2}{\sin. 1''}$ constant log.					5·615455
Reduction	136·9″			log.	2·146429

Reduc. to merid.		2′	17″	
True altitude	38	46	30	
Merid. altitude	38	48	47	
Mer. zen. dist.	51	11	13	S.
True dec.	4	10	26	S.
Latitude	55	21	39	S.

RULE FOR THE DIRECT METHOD WHEN AN OBJECT IS NEAR THE MERIDIAN.—(1) From the mean of the times of observation deduce the Greenwich time and hour angle.

(2) Correct the sun's declination and altitude.

(3) Then:—Log. tan. first arc = log. sec. hour angle + log. tan. corrected declination: to be named N. or S. according to the declination.

Log. cos. second arc = log. sin. first arc + log. cosec. corrected declination + log. sin. true altitude to be named of a contrary name to the bearing of the sun.

(4) If of the same name the sum of the arcs is the latitude, if of different names the difference of the arcs is the latitude, and is of the name of the greater.

Ex. 235. 1887, October 4th, a.m., at ship in latitude by account 55° 10′ S., longitude 140° 30′ E., the observed altitude of the sun's lower limb, bearing north, taken near the meridian was 38° 29′ 50″, and time by watch 9h. 47m. 14s. a.m. The index error was + 5′ 13″; height of eye 21 feet. The watch had been found to be slow 2h. 1m. 20s. of apparent time at ship, and the difference of longitude made to the eastward was $23\frac{1}{2}$ miles after the error on apparent time had been determined. Required the latitude.

The student will see that this is the same problem as has been already worked, hence we may take the data as there corrected.

Hour angle	=	2°	28′	0″
True altitude	=	38	46	30
Sun's declination	=	4	10	26 S.

For latitude by direct method.

Hour angle	2°	28′	0″	sec.	·000403
Declination	4	10	26	tan.	8·863187
First arc	4	10	40 S.	tan.	8·863590
First arc	4	10	40	sin.	8·862437
Declination	4	10	26	cosec.	1·137967
Altitude	38	46	30	sin.	9·796758
Second arc	51	10	56 S.	cos.	9·797162
First arc	4	10	40 S.		
Latitude	55	21	36 S.		

By referring to the example solved by "reduction to the meridian" it will be observed that the latitude agrees to 3 seconds.

TESTS OF ACCURACY OF WORK.—At sea it is quite unnecessary to work to seconds, because, whether we find the latitude by the reduction to the meridian, or by the direct method, there will be generally a trifling error in the hour angle. In the direct method, the second, third, fourth, and fifth logs. come all from one page of the tables, and the sixth and seventh from another, or nearly so. Again, the first arc is always a little greater than the declination, and the second arc is nearly the complement of the altitude when the object is near the meridian. All of these are such simple tests that no excuse for an error in them can be tolerated.

EXERCISE VIII.

Ex. 236. 1887, July 10th, a.m., at ship in latitude by account 53° 30′ N., longitude 2° 20′ W., the mean of a set of observed altitudes of the sun's lower limb was 60° 0′ 20″, bearing south. The mean of the times of observation by watch was 4h. 57m. 18s., which had been found fast the same morning 5h. 3m. 40s. of apparent time at ship, and the difference of longitude made to the westward since the error had been determined was 40·3 miles. If the index error of the sextant was − 14′ 33″, and height of the eye above the sea 18 feet, find the latitude.

Ex. 237. 1887, April 2nd, a.m., at ship in latitude by account 41° 3′ N., longitude 138° 4′ W., the mean of a set of altitudes of the sun's lower limb was 53° 17′ 40″, bearing south. Index error − 3′ 18″; height of eye 18 feet. The mean of the times of observation was 2h. 3m. 5s. by a watch which had been found to be fast of apparent time at ship the same morning 2h. 12m. 28s. The difference of longitude made by the ship since the error had been determined was 25 miles to the eastward. Find the latitude.

Ex. 238. 1887, November 29th, a.m., at ship in latitude by account 52° S., longitude 167° 1′ E. The mean of a set of altitudes of the sun's L. L. was 58° 23′, bearing north of the observer. Index correction for the sextant − 4′ 17″; height of the eye above the sea 19 feet. The mean of the times of observation was noon on the 29th by a watch which had been found to be fast 39m. 2s. of apparent time at ship on the same morning. The difference of longitude made to the westward after the error had been determined was 15 miles. Find the latitude.

Ex. 239. 1887, August 24th, in latitude by account 43° 45' S., longitude 23° 28' E., the altitudes of the sun near noon, bearing north, were taken at the following times to determine the latitude. Index error — 3' 18"; height of eye 25 feet.

Watch showed.			*Mean of observed alts.*
1h.	2m.	20s.	
1	3	10	
1	3	50	Sun's U. L. 35° 22' 54"
1	5	0	
1	6	20	

The watch had been found to be fast of apparent time at ship 1h. 15m. 42s., and the difference of longitude made to the westward was 10·2' after the error had been determined.

Ex. 240. 1887, on December 22nd, in latitude by account 39° 30' N., longitude 139° 20' E., the following observations were taken with an artificial horizon near noon, the sun bearing south. Index error of the sextant was + 23". The watch had been found slow of apparent time at place 25m. 53s., and the difference of longitude made to the westward after the error had been determined was 19·7 miles. Required the latitude.

Watch showed.			*Altitudes of sun.*			
11h.	56m.	20s.	L. L.	53°	28'	30"
11	57	30	U. L.	54	31	10
11	58	50	L. L.	53	26	40
0	0	40	U. L.	54	30	10
0	2	10	L. L.	53	23	50
0	3	50	U. L.	54	27	0

Ex. 241. If, on April 5th, 1887, in latitude by account 45° S., longitude 107° 20' 30" W., the following observations were taken near noon, the sun bearing north—index error of sextant — 2' 30"; height of the eye above the sea 20 feet—the watch had been found to be fast 1h. 29m. 15s. for apparent time at ship, and the difference of longitude made to the eastward after the error of the watch had been determined was 27·6 miles—required the latitude.

Watch showed.			*Altitudes sun's L. L.*		
0h.	59m.	23s.	38°	15'	20"
1	0	40	38	16	30
1	2	17	38	17	50
1	4	0	38	18	50

Ex. 242. 1887, September 25th. Mean time at ship 11h. 25m. p.m., in latitude by account 29° 10′ S., longitude 7° 29′ E., the altitude of Algenib by artificial horizon was 91° 57′ 30″, bearing N. Index-error + 1′ 22; height of eye 19 feet. The watch had been found to be fast of mean time at ship 2m. 18s. on the same afternoon, and the difference of longitude made to the eastward after the error had been determined was 12·7′. Find the latitude.

Ex. 243. 1887, May 12th, in latitude by account 59° 30′ S., longitude 14′ 30″ W., the mean time by a watch when an observation of the moon's L. L. was taken was 5h. 56m. a.m., the altitude being 48° 40′ zenith south of the moon. Index correction of the sextant + 3′ 1″; height of eye above the sea 22 feet. The watch had been found 1h. 22m. 17s. fast of ship mean time, and the difference of longitude made to the westward after the error of the watch had been determined was 10·5 miles. Required the latitude.

Ex. 244. State clearly the difference between finding latitude by "reduction to the meridian," and by the "direct method" in circum-meridional altitudes.

Ex. 245. Why is *apparent time* used in finding latitude by an exmeridian altitude of the sun, and how is apparent time at place found?

Ex. 246. Show why exmeridian altitudes for latitude are taken near the transits of celestial bodies.

Ex. 247. Show how to find the latitude by an altitude of the sun near the meridian without using the estimated latitude. *A.* 1873.

Ex. 248. Investigate a method for finding latitude by the "reduction to the meridian." *Royal Naval College*, 1874.

Ex. 249. In determining the latitude from the altitude of a heavenly body near the meridian, given a small error in the hour angle or time of the ship at sea: find the corresponding error in the latitude deduced from it. *A.* 1866.

Ex. 250. Explain clearly the ordinary method of determining, approximately, the latitude, at sea, by means of two or more altitudes of the sun taken near the meridian of the place of observation. *Second B.Sc. London*, 1878.

CHAPTER X.

On finding latitude by an altitude of Polaris—Proof of formulæ—Rigid formula—Proof for approximate formula—Rules—Examples—Exercise—Examination.

To find the Latitude by the Pole Star.—Latitude by the pole star consists of deducing the altitude of the pole (which has already been proved equal to the latitude), and can only be obtained north of the equator; but as the altitude of the star can be taken at any part of the night, except in very low latitudes, its application to the problem of determining latitude deserves particular attention. In the method of circum-meridional altitudes we have proved that to obtain correct results the azimuth of a body must be small when it is observed; and, as this star is so near to the pole, its azimuth must always be less than one-eighth of a point, and it is therefore particularly well adapted for finding latitude by that method. Obtaining latitude by the pole star possesses one great advantage over circum-meridional altitudes; viz. that as its motion is always so slow, owing to its polar distance being so small, an error in the estimated time has little influence on the result; an error of 5° of longitude even when the star is in its worst position for the purpose, i.e. when moving vertically, will affect the latitude only about 7 minutes.

This problem may be solved entirely by the method used in the last article, but owing to its polar distance being so small, and therefore the difference between the colatitude and zenith distance being also small, an independent formula is easily obtained by expanding the sine and cosine of these small angles in the usual series, viz.:—

$$\sin.\ \theta = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \&c.,$$

$$\text{and } \cos.\ \theta = 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \&c.$$

To find an expression for the Altitude of the Pole at any time. When Polaris is on the meridian below the pole, its polar distance, which is at once ascertained from the declination given in the "Nautical Almanac," added to the altitude will give the altitude of the pole or the latitude of the place.

When it is on the meridian above the pole, the polar distance subtracted from the altitude of the star will give the latitude.

When out of the meridian the following is the expression for latitude:—

$$l = a \pm p \,.\, \cos. h + \tfrac{1}{2} \sin. 1''. (p \,.\, \sin. h)^2. \tan. a$$
$$- \tfrac{1}{3} \sin.\, ^2 1''. (p \,.\, \cos. h)\, (p \,.\, \sin. h)^2.$$

Where l denotes the latitude of the place,
a „ „ altitude of polaris,
p „ its polar distance,
h „ „ hour angle.

Let l' be the colatitude of the place, and z be the zenith distance of the star, and let the difference between the colatitude and the zenith distance $= x$; then

$$x = l' - z = (90 - l) - (90 - a)$$
$$= \text{alt.} - \text{lat.} \quad . \quad . \quad . \quad . \quad . \quad \text{I.},$$

and x and p are both small quantities.

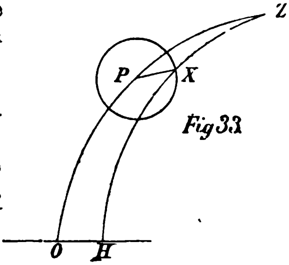


Fig 33

Let $H\,O$ be the horizon, Z the zenith, P the pole, X Polaris on a circle of declination, then $Z\,P\,X$ is the hour angle of the star; and

$$\cos. z = \cos. l'. \cos. p + \sin. l'. \sin. p \,.\, \cos. h$$
$$= \cos. (z + x) \cos. p + \sin. (z + x) \sin. p \,.\, \cos. h$$
$$= (\cos. z \,.\, \cos. x - \sin. z \,.\, \sin. x) \cos. p$$
$$+ (\sin. z \,.\, \cos. x + \cos. z \,.\, \sin. x) \sin. p \,.\, \cos. h$$
$$= \left\{ \cos. z \left(1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \&c.\right) - \sin. z \left(x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \&c.\right) \right\} \cos. p$$
$$+ \left\{ \sin. z \left(1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \&c.\right) + \cos. z \left(x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \&c.\right) \right\}$$
$$\times \sin. p \,.\, \cos. h$$
$$= (\cos. z - x \sin. z - \tfrac{1}{2} x^2 \cos. z + \tfrac{1}{6} x^3 \sin. z + \&c.) \cos. p$$
$$+ (\sin. z + x \,.\, \cos. z - \tfrac{1}{2} x^2 \,.\, \sin. z - \tfrac{1}{6} x^3 \,.\, \cos. z) \sin. p \,.\, \cos. h$$

$$= (\cos. z - x \sin. z - \tfrac{1}{2}x^2 \cos. z + \tfrac{1}{6}x^3 \sin. z + \&c.)\,(1 - \frac{p^2}{\lfloor 2} + \&c.)$$

$$+ (\sin. z + x . \cos. z - \tfrac{1}{2}x^2 \sin. z + \&c.)\,(p - \frac{p^3}{\lfloor 3} + \&c.) \cos. h.$$

Dividing by sin. z and neglecting the fourth and higher powers—

$$\left.\begin{array}{l}\text{Cot. } z = \cot. z - x - \tfrac{1}{2}x^2 \cot. z + \tfrac{1}{6}x^3 + \&c. \\ \quad - \tfrac{1}{2}p^2 \cot. z + \tfrac{1}{2}p^2x + \tfrac{1}{4}p^2x^2 \cot. z - \&c. \\ \quad + p . \cos. h + px \cot. z . \cos. h - \tfrac{1}{2}px^2 \cos. h + \&c. \\ \quad - \tfrac{1}{6}p^3 . \cos. h - \tfrac{1}{6}p^3x \cot. z . \cos. h + \&c.\end{array}\right\}$$

$$= \cot. z - x + p . \cos. h - \tfrac{1}{2}x^2 . \cot. z - \tfrac{1}{2}p^2 \cot. z$$
$$+ px . \cot. z . \cos. h$$
$$+ \tfrac{1}{6}x^3 + \tfrac{1}{2}p^2x - \tfrac{1}{2}px^2 \cos. h - \tfrac{1}{6}p^3 . \cos. h + \&c.$$

$$\therefore\ x = p . \cos. h - \tfrac{1}{2} \cot. z\,(x^2 + p^2 - 2px . \cos. h) +$$
$$\tfrac{1}{6}\,(x^3 + 3p^2x - 3px^2 \cos. h - p^3 . \cos. h)\ \&c.$$

Hence $x = p . \cos . h$ is the first approximation.

Substituting this value of x in the terms of the second order—

$$x = p . \cos. h - \tfrac{1}{2} \cot. z\,(p^2 . \cos.^2 h + p^2 - 2p^2 . \cos.^2 h)$$
$$= p . \cos. h - \tfrac{1}{2}p^2 \cot. z \sin.^2 h \text{ as a second approximation.}$$

Substituting the value of x now found in the terms of the second order, and $x = p . \cos. h$ in the terms of the third order, we get

$$x = p . \cos. h - \tfrac{1}{2}p^2 . \cot. z . \sin.^2 h + \tfrac{1}{3}p^3 . \cos. h . \sin.^2 h,$$

and if x and p be both expressed in seconds we have

$$x = p.\cos.h - \tfrac{1}{2}\sin. 1''.\cot.z(p.\sin.h)^2 + \tfrac{1}{3}\sin.^2 1''(p.\cos.h)(p.\sin.h)^2.$$

The value of the last term deduced from $u = \cos. h . \sin.^2 h$ when u is a maximum corresponds to $3 \sin.^2 h = 2$, and its value is then $\frac{2}{9\sqrt{3}}p^3. \sin.^2 1''$. Now as p at present is only about 1° 18′, the value of this term is less than half a second, and may therefore be neglected.

Then $x = p . \cos. h - \tfrac{1}{2} \sin. 1'' \cot. z\,(p . \sin. h)^2$.

But from I. lat. = alt. $- x$.

$\therefore l = a - p . \cos. h + \tfrac{1}{2} \sin. 1'' \tan. a\,(p . \sin. h)^2.$

In the "Nautical Almanac," pp. 477-9, we find three tables "*used in determining the latitude by observation of the pole star out of the meridian,*" and because p is constant and h depends on the sidereal time, therefore sidereal time is alone the argument, and Table I. is calculated from $p . \cos. h$, assuming $p = 78'\ 0''$.

In Table II. the altitude of the star and sidereal time are the arguments, and is calculated from $\frac{1}{2}$ sin. 1″ tan. a $(p \,.\, \sin.\, h)^2$ when $p = 78'\ 0''$ and $a = 19^\circ\ 15'$.

In Table III., which is special for the year, the correction depends on the difference between the true and assumed values of p and a. It has the sidereal time and date for the arguments, and is increased by 1′ for the purpose of rendering the quantities additive.

SECOND PROOF.—The following, not being so general, is a simpler method of finding the first two corrections :—

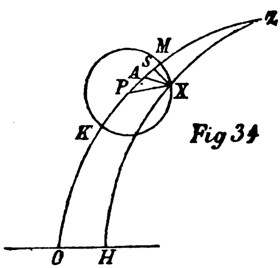


Fig 34

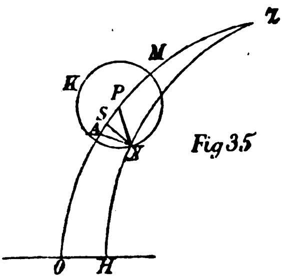


Fig 35

From X drop a perpendicular $X\,S$ on $Z\,O$, and make $P\,A$ the difference between the colatitude and the zenith distance, or what is the same thing, between the latitude and the altitude of the star.

$$\begin{aligned}
\text{Then latitude} &= O\,A \pm A\,P, \\
\text{but } A\,P &= S\,P \pm A\,S \\
\therefore \text{ latitude} &= O\,A \pm (S\,P \pm A\,S) \\
&= O\,A \pm S\,P + A\,S \\
&= a \pm p \,.\, \cos.\, h + A\,S \quad \ldots\ldots \quad \text{I.}
\end{aligned}$$

In the right-angled triangle $Z\,S\,X$

$$\text{Cos. } Z\,X = \cos.\, Z\,S \,.\, \cos.\, S\,X, \text{ but } Z\,X = Z\,A = Z\,S + A\,S.$$

$$\therefore \cos.\,(Z\,S + A\,S) = \cos.\, Z\,S \,.\, \cos.\, S\,X.$$

$$\text{Cos. } Z\,S \,.\, \cos.\, A\,S - \sin.\, S\,Z \,.\, \sin.\, A\,S = \cos.\, Z\,S \,.\, \cos.\, S\,X.$$

And as $A\,S$ is very small this may be written—

$$\text{Cos. } Z\,S - \sin.\, Z\,S \,.\, \sin.\, A\,S = \cos.\, Z\,S \,.\, \cos.\, S\,X$$

$$\sin.\, Z\,S \,.\, \sin.\, A\,S = \cos.\, Z\,S\,(1 - \cos.\, S\,X)$$

$$\sin.\, A\,S = \cot.\, Z\,S\left(2 \,.\, \sin.^2 \frac{S\,X}{2}\right);$$

but ZS is so nearly ZA, and AS and SX are such small quantities we may write the last equation—

$$AS''. \sin. 1'' = 2\left(\frac{SX''}{2}\right)^2 \sin.^2 1''. \cot. ZA.$$

$$\therefore AS'' = \tfrac{1}{2}(SX'')^2 \sin. 1'' \cot. ZX,$$

$$\text{and } SX = p \sin. h.$$

Hence $AS'' = \frac{1}{2} \sin. 1'' (p . \sin. h)^2 \tan. a$, and from I.—

$$\text{latitude} = a \pm p . \cos. h + \tfrac{1}{2} \sin. 1''. \tan. a (p . \sin. h)^2.$$

In explanation of the $\pm$ before $p . \cos. h$, this is deduced from the figures, and depends on the magnitude of h: and the term must be added when the star is below the pole, that is when the hour angle is between 6 and 18 hours, but must be subtracted when between 18 hours and 6 hours. The third term $\frac{1}{2}(p . \sin. h)^2 \sin. 1'' \tan. a$ is always added.

The following is the rule to be observed:—

RULE.—(1) Find the correct Greenwich date.

(2) Find the sidereal time by adding the R. A. of the mean sun to the mean time at place.

(3) Correct the altitude and subtract 1'.

(4) From Table I. ("Nautical Almanac") take out the correction and apply as there directed.

(5) With the altitude and the sidereal time take the second correction from Table II. and add it.

(6) With the sidereal time and the day of the month take the correction from Table III. and add. The result is the latitude.

In (3) the 1' is subtracted from the altitude so as to make the third correction always additive; and in some years, e.g. 1865, it is necessary to subtract 2'; but this can always be seen in the "Nautical Almanac."

At sea all seconds may be omitted, and the first correction $p . \cos. h$ need only be applied, as the result will then be less than 2' in error in all latitudes from 0 to 60° N. Now $p . \cos. h$ can be taken direct from the traverse table where $p = 78$, and h is the number of degrees in the hour angle.

To solve this problem without the aid of the special tables given in the "Nautical Almanac" the following rule should be observed:—

RULE—(1) Find the correct Greenwich date and correct the altitude.

(2) Take out the R. A. and declination of Polaris for the Greenwich date, find the polar distance and reduce it to seconds (p).

(3) Correct the R. A. of the mean sun, and to it add the mean time at place; this is the sidereal time at place. From the sum subtract the R. A. of Polaris; the remainder is the star's hour angle. Reduce this to degrees, minutes. and seconds.

(4) Find $p \cdot \cos. h$. Add log. p to log. cos. h, and reduce the seconds to minutes, &c. If the sidereal time at ship be between 6 hours and 18 hours, add $p \cdot \cos. h$, otherwise it must be subtracted.

(5) Find $\frac{1}{2}$ sin. $1''$ $(p \cdot \sin. h)^2$. tan. a. To log. p add log. sin. h, double this and add log. tan. altitude and the constant log. 4·384545 ($\frac{1}{2}$ sin. $1''$) to the product. The result is the second correction always to be added.

Ex. 251. 1887, August 24th, at 12h. 15m. 25s. mean time at ship, the observed altitude of Polaris was 59° 25′ 10″ in longitude 27° 13′ 40″ W. Index error + 3′ 19″; height of the eye above the sea 21 feet. Find the latitude.

For Greenwich date.

Mean time at place, Aug.	24d.	12h.	15m.	25s.
Long. in time, W.		1	48	55
Mean time Greenwich, Aug.	24	14	4	20

Long. in time.

Long. 27° 13′ 40″ W.
4

60) 108 54 40

1h. 48m. 55s.

For hour angle.

Sidereal time, Aug. 24	10h.	9m.	47·44s.
Acceleration for 14h.		2	17·99
Acceleration for 4m.			0·66
Acceleration for 20s.			0·05
R. A. mean sun	10	12	6·14
Mean time place	12	15	25·00
R. A. meridian	22	27	31·14 = sidereal time at place.
R. A. Polaris	1	18	24·34
W. hour angle	21	9	6·80
E. hour angle	2	50	53·2 = 42° 43′ 18″
Observed alt.	59	25	10
Index error		+ 3	19
	59	28	29
Dip (21 feet)		4	31
	59	23	58
Refraction			34
True alt.	59	23	24

Data for Polaris.

R. A.	1h. 18m. 24·34s.
Declination	88° 42′ 13·3″
P. D.	1 17 46·7
	= 4666·7″

For latitude.

By "Nautical Almanac."

True alt. =	59°	23′	24″
Constant =	− 1		0
Reduced alt.	59	22	24
First cor.	− 57		37
Approx. lat.	58	24	47
Second cor. +			42
Third cor. +		1	28
Latitude	58	26	57 N.

By formulæ.

First correction p. cos. h.

$p = 4666{\cdot}7$	log.	3·669010
$h = 42° \; 43' \; 18''$	cos.	9·866085
3428″	log.	3·535095
First correction	= −	57′ 8″

Second cor. $= \frac{1}{2}\,\textit{sin.}\,1''\,(p\,.\sin h)^2\,.\,\textit{tan.}\,a.$

$p = 4666{\cdot}7$	log.	3·669010
$h = 42° \; 43' \; 18''$	sin.	9·831510
		3·500520
		2
$(p\,.\,\sin.\,h)^2$		7·001040
$a = 59° \; 23' \; 24''$	tan.	·227947
Constant $\frac{1}{2}$ sin. 1″		4·384545
41·07″	log.	1·613532

Altitude Polaris	59° 23′ 24″
First correction	— 57 8
	58 26 16
Second correction	+ 41
Latitude	58 26 57 N.

EXERCISE IX.

Ex. 252. 1887, January 7th. The observed altitude of the pole star in longitude 133° 17′ 30″ W. at 3h. 27m. 10s. a.m. mean time, was 49° 2′ 10″. The index error of the sextant was + 1′ 17″; height of eye above the sea 24 feet. Required the latitude.

Ex. 253. 1887, November 5th, at 13h. 30m. 30s. mean time at place in longitude 169° 23′ E., the observed altitude of Polaris was 53° 25′ 15″. Height of the eye 20 feet; index error — 3′ 25″. Required the latitude.

Ex. 254. 1887, February 14th, in longitude 7° 29′ E. at 10h. 54m. 29s. mean time at ship, the observed altitude of the pole star was 65° 58′ 45″. Index correction for the sextant + 2′ 23″; height of the eye above the sea 18 feet. Find the latitude.

Ex. 255. 1887, December 16th, at 7h. 33m. mean time at

ship in longitude 125° 39′ W., the observed altitude of the star α Ursæ Minoris was 38° 22′ 10″. Index error — 3′ 15″; Height of the eye above the sea 25 feet. Required the latitude.

Ex. 256. 1887, July 15th, mean time at ship 9h. 22m. 10s. in longitude 47° 29′ 30″ W., the observed altitude of the star α Ursæ Minoris was 41° 25′ 10″. Index error — 7′ 15″; height of eye 23 feet. Find the latitude.

Ex. 257. How do you recognize Polaris in the heavens? Why does that star remain stationary while the other stars appear in different parts of the heavens at different times of the night?

The declination of Regulus is 12° 32′ N.; in what latitude is it just visible at its lower culmination? *E.* 1882.

Ex. 258. Show that the elevation of the pole at any time is the latitude of the observer; and illustrate by a diagram when latitude can be found from the altitude of the pole star and its declination only.

Ex. 259. Explain how it is that the pole star furnishes a convenient method for determining latitude. Can this method be used in every part of the world, and why? July 31st, 1887, at 9h. 24m. p.m., in longitude 1° 25′ W., the observed altitude of Polaris was 54° 49′. Index correction — 30″; height of eye 30 feet. Required latitude. *A.* 1879.

Ex. 260. How is it that the pole star is so useful in determining the latitude? Investigate the rule for finding the latitude by the altitude of the pole star. *A.* 1874.

Ex. 261. What objection is there to using the formula proved in the last question in determining the latitude by other circumpolar stars?

Ex. 262. Explain by construction the following rules:—

(*a*) Latitude by meridian altitude below pole.
(*b*) „ „ pole star.
(*c*) „ „ altitude near the meridian.

Royal Naval College, 1864.

Ex. 263. Determine the latitude by an observation of Polaris from the formula—

$$l = a - p \,.\, \cos. h + \tfrac{1}{2} \sin. 1'' \, (p \,.\, \sin. h)^2 \,.\, \tan. a$$
$$- \tfrac{1}{3} \sin.^2 1'' \, (p \,.\, \cos. h) \, (p \,.\, \sin. h)^2$$

when
$$a = 32° \, 36' \, 44''$$
$$p = 5256''$$
$$h = 3\text{h. } 50\text{m. } 10\text{s.}$$

A. 1873.

Ex. 264. Explain the construction of Inman's table for "correction of pole star." *For Beaufort Testimonial*, 1864.

Ex. 265. Show :—

$$\text{Latitude} = a \pm p \,.\, \cos.\, h + \tfrac{1}{2} \sin.\, 1''.\, (p \,.\, \sin.\, h)^2 \tan.\, a$$
$$- \tfrac{1}{3} \sin.\, ^2 1'' \,(p \,.\, \cos.\, h)\,(p \,.\, \sin.\, h)^2.$$

Compute the maximum value of the last term if $p = 87'$, and hence show that in practice it may be neglected.

Ex. 266. Explain by help of a diagram the principle of finding the latitude by altitude of Polaris, and show how the table "Correction for pole star" is computed.

Construct a table for the year 1880, latitudes 50° and 70°, and R. A. of meridian 8h. 30m. *H.* 1880.

Ex. 267. Explain clearly what "arguments" of tables are, writing down the formula from which the three tables in the "Nautical Almanac," pages 477-8-9, for finding the latitude by Polaris are computed. Explain how these tables are constructed.

Work the following example, obtaining the latitude approximately by the application of the first two corrections, which are required to be obtained from the formula without the help of the tables.

April 20th, 1887, at 1h. 30m. a.m. mean time nearly, in longitude 37° 30′ W., the observed altitude of Polaris was 29° 15′ 20″, the index error +1′ 30″, height of eye 18 feet. Required the latitude. *H.* 1877.

Ex. 268. Obtain and investigate the methods of finding the latitude by circummeridian altitudes; (1) using approximate latitude, (2) independent of the approximate latitude. Examine the case of the pole star. *H.* 1883.

Ex. 269. Describe some method by which the latitude of a place may be found, and mention the names of the instruments which are required to make the necessary observations. What method would be used at sea?

Second B.A. and B.Sc. London, 1869.

CHAPTER XI.

On longitude—Recommendations of the International Geodic Association—The chronometer as a means for finding mean time at Greenwich—On finding the hour angle of an object—Proof of formulæ—For a small error in altitude to find the corresponding error in the hour angle—For a small error in latitude to find the corresponding error in the hour angle—The position of an object when its change in altitude is a maximum—On the most favourable position of an object for finding longitude from its altitude—Directions for observing—Rules—Examples—Exercise—Examination.

Having shown how to deduce latitude, one of the co-ordinates for fixing the position of a place on the earth's surface, it now devolves on us to show how longitude, the other co-ordinate, is found. The want of sufficiently accurate methods for determining longitude was acutely felt in England when we were extending our empire by discovery and conquest under the Tudors and Stuarts; and in recognition of that want Charles II., in the year 1675, founded the Royal Observatory at Greenwich, and appointed Flamstead the first Astronomer-Royal. The words of his commission were: "*To apply himself with the utmost care and diligence to the rectifying the tables of the motions of the heavens, and the places of the fixed stars, in order to find out the so much desired longitude at sea, for perfecting the art of navigation.*"

Longitude of a place is the angle at the axis of the earth between two planes, one through the first meridian, the other through the meridian of the place; or

Longitude is the angle at the poles between two meridians, one through the place of reference, the other through the given place; or

Longitude is the angle at the centre of the earth subtended by an arc of the equator intercepted between the first meridian and the meridian through the place.

Unlike latitude, which has a fixed great circle, the equator, from which all nations agree to measure, longitude has none;

but each nation takes as its first meridian that one which passes through its national observatory or chief town. English-speaking peoples have agreed to accept that through Greenwich as the first meridian, and longitude by them is at present reckoned from the meridian of Greenwich 180° E. and 180° W. At the conference of the International Geodic Association, held at Rome in October, 1883, which was attended by delegates from all the principal maritime nations in the world, it was, after an animated debate, agreed:—

"*Firstly*, that the unification of longitudes and hours is as equally desirable in the interests of science as in those of navigation, commerce, and international communication.

"*Secondly*, that the conference propose to the Governments to choose for the initial meridian that of Greenwich, inasmuch as that meridian fulfils, as a point of departure of longitudes, all the conditions required by science.

"*Thirdly*, that the longitudes should be reckoned from the meridian of Greenwich in the sole direction of from east to west, and from zero to 360°, or from zero to twenty-four hours.

"*Fourthly*, that the conference recognizes for certain scientific needs, &c., the utility of adopting a universal hour, side by side with the local or national hours, which will necessarily continue to be employed in civil life.

"*Fifthly*, that the conference recommends, as the point of departure of the universal hour and cosmopolitan dates, the mean noon of Greenwich, which coincides with the instant of midnight or with the beginning of the civil day, situate at the twelfth hour, or at 180° Greenwich.

"*Sixthly* and *seventhly*, recommend that the Governments of the different states in the world should adopt these suggestions as soon as possible, and put them into practical effect." This is a recommendation which all scientific men and practical navigators will devoutly echo.

Because the earth rotates on its axis in twenty-four hours, therefore every point on its surface describes a complete circle in that time; and because her rotation is uniform, the angles described by every such point in equal times are equal. As the earth rotates from west to east, the sun must cross the meridian of a place to the eastward of another before it crosses the meridian of the westerly place. The place to the eastward will therefore have the later time of the two, and hence, if the time at place be later than at Greenwich, the place is in east longi-

tude; but if earlier it must be in west longitude. From the above we may consider the longitude of a place as the interval of mean time which elapses between the sun crossing the meridian of Greenwich and the meridian of the place in question. Longitude, therefore, as first pointed out by Gemma Frisius, in 1545, being measured by the difference of times between the place and at Greenwich, the determination of longitude resolves itself into—

(1) Finding the mean time at Greenwich.

(2) Finding the mean time at the place of observation.

The first was for a great number of years justly considered the problem of the age, and many of the greatest men of the seventeenth and eighteenth centuries bent their energies towards its solution. Such men were Hooke, in our own country, who invented balance-springs for watches and the anchor escapement; Huyghens, in Holland, who first constructed clocks with pendulums; and Le Roy, father and son, and Berthord, in France. But until John Harrison constructed his chronometer little advance, if we except Le Roy's timekeeper, was made. Stimulated by the offer of 20,000*l.* made by the British Parliament, in 1714, for any method by which longitude could be determined within thirty miles, Harrison invented gridiron pendulums for clocks, and compensation balance wheels for watches; and he brought his inventions to such perfection that he was, after long delay and repeated trials, awarded the promised amount. The chronometer, as perfected by Arnold, Mudge, Evenshaw, Brooksbank, and others, is a superior kind of watch, whose rate remains the same under all meteorological conditions. Heat and cold do not affect its working in any material degree, and the Greenwich mean time can, whenever required, be deduced from its original error and rate. The method to be employed has already been fully explained, pp. 40 and 41.

About the same time as the improvements in timekeepers were engaging so much attention in all maritime nations, tables of the exact positions of the heavenly bodies were also recalculated by Professor Mayer, of Gottingen, whose widow received 3000*l.* from the British Parliament for her husband's labours. A sum of 300*l.* was also awarded to Euler, in consideration that Mayer had made use of his lunar theory. At the recommendation of the Board of Longitude, all tables of the position of celestial objects necessary for computing longitude were, in 1766, ordered to be printed under the superintendence of the then Astronomer-Royal, Dr. Nevil Maskelyne. The work

was entitled, "A Nautical Almanac and Astronomical Ephemeris," and it has, with many additions from time to time, been continued annually, and is published, by order of the Lords Commissioners of the Admiralty, three or four years in advance, for the convenience of mariners and astronomers.

The finding of mean time at place for longitude at sea resolves itself into calculating the hour angles of the heavenly bodies, and depends on the latitude of the observer and the altitude and declination of the selected body. Main, the late Ratcliffe observer at Oxford, says, "A great many different methods for finding longitude at sea have been employed; but it is the most difficult and at the same time vitally important of the problems which the mariner has to solve; but all the methods are different applications of the principle mentioned above." The rules for the computations of the hour angle are deduced as follows:—

PROOF OF RULE FOR FINDING THE HOUR ANGLE.

Fig 36.

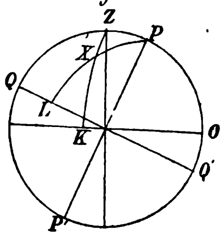
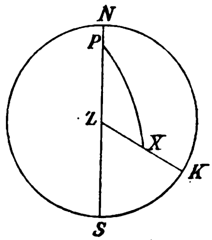


Fig 37.



In the figures, let X be the object out of the meridian, and the other letters denote the same as in the preceding articles;

$$\text{then } \cos. ZPX = \frac{\cos. ZX - \cos. PX . \cos. PZ}{\sin. PX . \sin. PZ};$$

or adopting the nomenclature already used,—

$$\cos. h = \frac{\cos. (90 - a) - \cos. p . \cos. (90 - l)}{\sin. p . \sin. (90 - l)};$$

$$\therefore 1 - 2 \sin.^2 \frac{h}{2} = \frac{\sin. a - \cos. p . \sin. l}{\sin. p . \cos. l}.$$

$$2 \sin.^2 \frac{h}{2} = 1 - \frac{\sin. a - \cos. p . \sin. l}{\sin. p . \cos. l}$$

$$= \frac{\sin. p . \cos. l + \cos. p . \sin. l - \sin. a}{\sin. p . \cos. l}$$

$$= \frac{\sin. (p + l) - \sin. a}{\sin. p . \cos. l}$$

$$= \frac{2 \cos. \frac{p + l + a}{2} . \sin. \frac{p + l - a}{2}}{\sin. p . \cos. l};$$

$$\therefore \sin. \frac{h}{2} = \sqrt{\sec. l . \operatorname{cosec.} p . \cos. \frac{p + l + a}{2} . \sin. \frac{p + l - a}{2}}. \quad (\mathrm{I})$$

Let $S = \frac{p + l + a}{2}$ and take logs. of each side;

then $\log. \sin. \frac{h}{2} = \frac{1}{2}\{\log. \sec. l + \log. \operatorname{cosec.} p + \log. \cos. S + \log. \sin. (S - a)\}$.

This is the form which Borda adopted in his treatise on the reflecting circle. Many other forms, may be deduced such as the following, which is found in some works:—

$$\cos. h = \frac{\cos. z - \cos. (90 - d) . \cos. (90 - l)}{\sin. (90 - d) . \sin. (90 - l)};$$

$$\therefore 1 - 2 \sin.^2 \frac{h}{2} = \frac{\cos. z - \sin. d . \sin. l}{\cos. d . \cos. l},$$

$$2 \sin.^2 \frac{h}{2} = 1 - \frac{\cos. z - \sin. d . \sin. l}{\cos. d . \cos. l}$$

$$= \frac{\cos. d . \cos. l + \sin. d . \sin. l - \cos. z}{\cos. d . \cos. l}$$

$$= \frac{\cos. (l - d) - \cos. z}{\cos. d . \cos. l}$$

$$2 \sin^2 \frac{h}{2} = 2 \sin. \frac{z + (l - d)}{2} . \sin. \frac{z - (l - d)}{2} . \sec. d . \sec. l \quad . \quad . \quad (\mathrm{A}) \quad (\mathrm{II})$$

Taking logarithms of each side,—

$$\log. \sin. \frac{h}{2} = \frac{1}{2} \left\{ \log. \sec. d + \log. \sec. l + \log. \sin. \frac{z + (l - d)}{2} + \log. \sin. \frac{z - (l - d)}{2} \right\}.$$

In the Royal Navy, where Inman's tables are used, containing a table of Haversines (half the Versed sine), the conclusion of the proof depends on the following:—

$$1 - \cos.\ A \text{ or Vers. } A = 2 \sin.^2 \frac{A}{2}\,;$$

$$\therefore\ 2 \text{ Hav. } A = 2 \sin.^2 \frac{A}{2},$$

$$\text{and } \sin.\ \frac{A}{2} = \sqrt{\text{Hav. } A}.$$

Then returning to the formula marked (A), viz.,—

$$2 \sin.^2 \frac{h}{2} = 2 \sin.\ \frac{z + (l - d)}{2} \,.\, \sin.\ \frac{z - (l - d)}{2} \,.\, \sec.\ d \,.\, \sec.\ l,$$ (I)

we get Hav. h

$$= \sec.\ d \,.\, \sec.\ l \,.\, \sqrt{\text{Hav. } \{z + (l - d)\}} \,.\, \sqrt{\text{Hav. } \{z - (l - d)\}}.$$ (B)

It will be seen that the proofs given above apply to finding the hour angle of *any* heavenly body. If that body be the sun, the hour angle is the apparent time from noon, and hence the apparent time at place is at once known. Mean time at place is found by the application of the equation of time. If the body be the moon, planet, or star, its hour angle *alone* gives us no information as to local time, but the mean time at place is deduced by referring the object and the sun to the "first point of Aries" by the formula, p. 94.

M. T. at place = hour angle + R. A. of object — R. A. mean sun. That is, to the hour angle of the body as calculated above, add its R. A.; this gives the R. A. of the meridian; from the sum subtract the R. A. of the mean sun, the result is mean time at the place of observation. Practical men should accustom themselves to finding longitude by other celestial objects than the sun, for it frequently happens that owing to clouds, rain, fog, and other causes, mariners are whole days without seeing either the sun or the horizon; but there are few nights when neither moon, planets, nor stars are visible for time enough to make an observation.

We have spoken of the altitude of the body; but in all problems treated of in Nautical Astronomy, instead of a single altitude, it will be best, whenever practicable, to observe a series of altitudes in quick succession, noting the times when taken. Then a mean of the altitudes should be considered as corresponding to the mean of the times: as by such a process any great error in the time or altitude can generally be detected by inspection; and the mean of several altitudes will be nearer the correct observed altitude than any single observation, because, the shifting of the horizon owing to the roll of the sea

and the motion of the ship will thus be counteracted. But it can be proved that the position of an object in the heavens will greatly modify the results if a small error of observation be made. This is shown as follows :—

On the most favourable Position of an Object for taking Observations for Time and Longitude.

A. *For a small difference in altitude find the corresponding change in the hour angle.*—The fundamental formula used throughout Nautical Astronomy is—

$\text{Cos. } z = \cos. p . \cos. l' + \sin. p . \sin. l' . \cos. h,$

when $z = ZX$ the zenith distance.

$p = PX$ the polar distance.

$l' = PZ$ the colatitude.

$h = ZPX$ the hour angle.

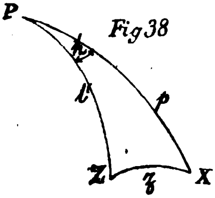


Fig 38

Differentiating the fundamental formula with respect to z and h—

$-\sin. z . dz = -\sin. p . \sin. l' . \sin. h . dh;$

$$\therefore\ dh = dz \frac{\sin. z}{\sin p . \sin. l' . \sin. h} \quad . \quad . \quad (a)$$

In the spherical triangle PXZ

$$\frac{\sin. XPZ}{\sin. PZX} \text{ or } \frac{\sin. h}{\sin. Z} = \frac{\sin. z}{\sin. p};$$

$$\therefore\ \sin. h = \frac{\sin. z . \sin. Z}{\sin. p}.$$

Substituting this value of sin. h in (a) we get

$$dh = dz \frac{\sin. z . \sin. p}{\sin. p . \sin. l' . \sin. Z \sin. z}$$

$$= dz \frac{1}{\sin. l' . \sin. Z}$$

where dh is the required small difference in the hour angle, dz the required small given difference in the zenith distance and therefore in the altitude, and Z the azimuth: hence small diff. in hr. $\angle = -$ small diff. in alt. $\times \frac{1}{\text{cos. lat. sin. azi.}}$. Now, as the latitude is considered constant, the small difference in the hour angle will vary inversely as sine azimuth, and will therefore be least when sine azimuth is greatest.

Therefore the small difference in the hour angle is least for

a small change in the altitude when the azimuth is 90°, or the object is on the prime vertical.

B. *For a small difference in the latitude find the corresponding change in the hour angle.*—Beginning with the fundamental formula

$$\text{Cos. } z = \cos. p \, . \cos. l' + \sin. p \, . \sin. l' \cos. h,$$

and differentiating with l' and h as variables,

$$o = -\cos. p \, . \sin. l' \, . \, dl' + \sin. p \cos. h \, . \cos. l' \, . \, dl' - \sin. p \, . \sin. l' \, . \sin. h \, . \, dh \, ;$$

$$\therefore \; dh = dl' \frac{\sin. p \, . \cos. l' \, . \cos. h - \cos. p \, . \sin. l'}{\sin. p \, . \sin. l' \, . \sin. h} \quad . \quad (b)$$

and dh vanishes when the numerator of the fraction on the right-hand side of the equation does so, i.e. when

$$\sin. p \, . \cos. l' \cos. h = \cos. p \, . \sin. l',$$
$$\text{or } \cos. h = \cot. p \, . \tan. l'.$$

Now this is the case only when the triangle PZX is right-angled at Z: hence the small error in the hour angle vanishes for a small difference in latitude when the object is on the prime vertical.

C. *To find the position of an object when its change in altitude is greatest in a given small time.*

$$\text{Cos. } p = \cos. l' \, . \cos. z + \sin. l' \, . \sin. z \cos. Z.$$

Differentiate with respect to z and Z—

$$o = -\cos. l' \, . \sin. z. \, dz + \sin. l' \, . \cos. Z \, . \cos. z \, . \, dz - \sin. l' \, . \sin. z \, . \sin. Z \, . \, dZ$$

$$\therefore \; dZ = dz \, . \frac{\sin. l' \cos. Z \, . \cos. z - \cos. l' \, . \sin. z}{\sin. l' \, . \sin. z \, . \sin. Z}$$

$$= dz \, (\cot. z \, . \cot. Z - \cot. l' \, . \operatorname{cosec.} Z)$$

$$\text{and } dz = \frac{dZ}{\cot. z \, . \cot. Z - \cot. l' \, . \operatorname{cosec.} Z} \quad . \quad . \quad . \quad . \quad (c)$$

where dZ is a small change in azimuth, and dz the corresponding small change in the zenith distance or altitude.

Now the denominator is least when $Z = 90°$.

Hence dz, or the small change in the zenith distance, is greatest when Z or azimuth is 90°, or the object is on the prime vertical, and therefore the altitude of an object changes fastest on the prime vertical.

Other Proofs.—For those who know trigonometry, but

have no knowledge of the differential calculus, we subjoin the following proofs of the above propositions :—

A. $\cos. z = \cos. p . \cos. l' + \sin. p . \sin. l' . \cos. h$. (1)

Let z take the small increment dz, then h will take a corresponding small increment dh, and p and l' are considered constant. Then—

$$\cos. (z + dz) = \cos. p . \cos. l' + \sin. p . \sin. l' . \cos. (h + dh) \quad (2)$$

Subtracting (2) from (1)—

$$\cos. z - \cos (z + dz) = \sin. p . \sin. l' \{\cos. h - \cos. (h + dh)\}.$$

$$\text{Cos. } z - \cos\ z . \cos. dz + \sin. z . \sin. dz$$
$$= \sin. p . \sin. l' (\cos. h - \cos. h . \cos. dh + \sin. h . \sin. dh),$$

and, because dz and dh are so small, we may take their cosines equal to unity.

$$\therefore \cos. z - \cos. z + \sin. z . \sin. dz$$
$$= \sin. p . \sin. l' (\cos. h - \cos. h + \sin. h . \sin. dh),$$

and $\sin. z . \sin. dz = \sin. h . \sin. p . \sin. l' . \sin. dh.$

Again, because dz and dh are considered very small, we may take

$$\sin. z = dz . \sin. 1'' \text{ and } \sin. dh = dh . \sin. 1'';$$

then $\sin. z . dz . \sin. 1'' = \sin. h . \sin. p . \sin. l' . dh \sin. 1''$;

$$\therefore dh = dz . \frac{\sin. z}{\sin. h . \sin. p . \sin. l'}.$$

This result corresponds with the equation marked (*a*), and the proof may be finished as before :—

B. $\cos. z = \cos. p . \cos. l' + \sin. p . \sin. l' . \cos. h$ (1)

If p and z be considered constant, then when the colatitude increases the hour angle will decrease, and *vice versâ ;* hence, when l' takes the increment dl', let h take the corresponding minus increment, or $(- dh)$,

then $\cos. z = \cos. p . \cos. (l' + dl') + \sin. p . \sin. (l' + dl') . \cos. (h - dh)$ . . . (2)

Subtracting (2) from (1)—

$$o = \cos.\ p\ \{\cos. l' - \cos. (l' + dl')\}$$
$$+ \sin.\ p\ \{\sin. l' . \cos. h - \sin (l' + dl') . \cos. (h - dh)\}$$

$$\sin. (l' + dl') \cos. (h - dh) - \sin. l' \cos. h$$
$$= \cot.\ p\ \{\cos. l' - \cos. (l' + dl')\}$$

$$(\sin. l' . \cos. dl' + \cos. l' . \sin. dl') (\cos. h . \cos. dh + \sin. h . \sin. dh)$$
$$- \sin. l' \cos. h$$
$$= \cot. p \{\cos. l' - \cos. l' \cos. dl' + \sin. l' . \sin. dl'\}.$$

By the hypothesis dl' and dh are very small, and therefore we may take their cosines equal to unity, then

$$(\sin. l' + \cos. l'. \sin. dl')(\cos. h + \sin. h . \sin. dh) - \sin. l' . \cos. h$$
$$= \cot. p \{\cos. l' - \cos. l' + \sin. l' . \sin. dl'\}.$$

Multiplying up

$$\sin. l' . \cos. h + \cos. l' \cos. h . \sin. dl' + \sin. l' . \sin. h . \sin. dh$$
$$+ \cos. l' . \sin. dl' . \sin. h . \sin. dh - \sin. l' . \cos. h$$
$$= \cot. p . \sin. l' . \sin. dl'.$$

Now $\cos. l'$ and $\sin. h$ are each proper fractions, and $\sin. dl'$ and $\sin. dh$ are each very small, so that the term

$$\cos. l' . \sin. dl' . \sin. h . \sin. dh$$

may be neglected without introducing any material error into the calculation. Then

$$\sin. l' . \sin. h . \sin. dh = \sin. dl' (\cot. p . \sin. l' - \cos. l' . \cos. h).$$

Again, $\sin. dh = dh . \sin. 1''$ and $\sin. dl' = dl' . \sin. 1''$;

$$\therefore\ dh . \sin. 1'' = dl' \sin. 1'' \frac{\cot. p . \sin. l' - \cos. l' . \cos. h}{\sin. l' . \sin. h},$$

$$\text{or } dh = -dl' \frac{\sin. p . \cos. l' . \cos. h - \cos. p . \sin. l'}{\sin. p . \sin. l' . \sin. h}.$$

The $-$ sign shows that if the colatitude be taken too great, then the hour angle will be too small. The formula then agrees with that marked (b), and the proof may be completed as before—

C. $\quad \text{Cos}. p = \cos. l' . \cos. z + \sin. l' . \sin. z . \cos. Z \quad . \ (1)$

Let dz and dZ be the corresponding increments of z and Z, so that $\cos. p = \cos. l' \cos. (z + dz)$

$$+ \sin. l' . \sin. (z + dz) . \cos. (Z + dZ) \quad . \quad . \quad . \ (2)$$

Subtracting (2) from (1)—

$$o = \cos. l' \{\cos. z - \cos. (z + dz)\}$$
$$+ \sin. l' \{\sin. z . \cos. Z - \sin. (z + dz) \cos. (Z + dZ)\}$$
$$\therefore\ \sin. (z + dz) . \cos. (Z + dZ) - \sin. z . \cos. Z$$
$$= \cot. l' \{\cos. z - \cos. (z + dz)\}$$

Expanding

$$(\sin. z . \cos. dz + \cos. z . \sin. dz)(\cos. Z . \cos. dZ - \sin. Z . \sin. dZ)$$
$$- \sin. z . \cos. Z = \cot. l' \{\cos. z - \cos. z . \cos. dz + \sin. z . \sin. dz\}$$

dz and dZ are each considered so small that their cosines may be taken equal to unity.

$$\therefore (\sin. z + \cos. z . \sin. dz)(\cos. Z - \sin. Z . \sin. dZ) - \sin. z . \cos. Z$$
$$= \cot. l' \{\cos. z - \cos. z + \sin. z . \sin. dz\}$$

Multiplying up

$$\sin. z \,.\, \cos. Z + \cos. Z \,.\, \cos. z \,.\, \sin. dz - \sin. z \,.\, \sin. Z \,.\, \sin. dZ - \cos. z \,.\, \sin. dz \,.\, \sin. Z \,.\, \sin. dZ - \sin. z \,.\, \cos. Z = \cot. l' \,.\, \sin. z \,.\, \sin. dz.$$

Cos. z and sin. Z are each proper fractions, and sin. dz and sin. dZ are each so small, that no material error will be introduced by neglecting the term containing the product of these four quantities; then

$$\sin. dz\,(\cos. Z \,.\, \cos. z - \cot. l' \,.\, \sin. z) = \sin. z \,.\, \sin. Z \,.\, \sin. dZ.$$

Again, $\sin. dz = dz \,.\, \sin. 1''$ and $\sin. dZ = dZ \,.\, \sin. 1''$;

$$\therefore\ dz \,.\, \sin. 1'' = dZ \,.\, \sin. 1'' \frac{\sin. z \,.\, \sin. Z}{\cos. Z \,.\, \cos. z - \cot. l' \sin. z},$$

$$\text{and } dz = dZ \frac{1}{\cot. z \,.\, \cot. Z - \cot. l' \operatorname{cosec.} Z},$$

which is the same formula as that marked (c); the proof is completed as before.

Hence—

I. Because the error in the hour angle, corresponding to a small error in the altitude, is least when the object is on the prime vertical.

II. Because the error in the hour angle, corresponding to a small error in latitude, vanishes when the object is on the prime vertical; and

III. Because the object rises or falls fastest when it is on the prime vertical, and its altitude can then be taken with greater precision :—

Therefore the most favourable position of an object for taking observations for time or longitude is when it bears due east or west, provided the altitude be then greater than 12°; because if the object be lower than 12°, the correction for refraction is so uncertain that it cannot be depended upon.

Directions for Observing.—(*a*) The object should be as near the prime vertical as possible if its altitude be not less than 12°.

(*b*) When between the tropics, and the latitude and declination are of the same name, the proximity of the *sun* to the meridian will have but little effect on the hour angle, because his motion in altitude is then sufficiently rapid to ensure correct results.

(*c*) Five or six altitudes should be taken in quick succession with the corresponding times, and the means of these should be used as the observed altitude and the observed time.

(*d*) If any other object can be used for calculating the hour angle, the moon should never be selected:—

(1) Because her declination changes so rapidly, often amounting to two minutes of arc in ten minutes of time.

(2) Because her right ascension changes so quickly, that often one minute error in Greenwich mean time will cause two seconds in error in the moon's right ascension.

And thus small errors may be introduced, which could not be the case if other celestial bodies be used for the purpose.

(*e*) If possible, longitude from equal altitudes of an object on each side of the meridian should be computed. The mean of these will give more nearly correct results than a single observation; because any error in the altitude or latitude will cause an error in the hour angle of different names on different sides of the meridian.

(*f*) It will be found that at twilight both stars and horizon may be distinctly seen, so that an altitude may easily be obtained.

The rule for computing longitude is as follows:—

To find longitude by a Sun Chronometer.

RULE (*a*) Find from the chronometer the correct Greenwich date.

pag 145(I).

(*b*) Correct the declination and equation of time from page II. of the "Nautical Almanac," and find the polar distance.

(*c*) Correct the altitude.

(*d*) Place altitude (a), latitude (l), and polar distance (p) one under the other: find their sum and half-sum (S), and from the latter subtract the altitude ($S-a$). Add together the logarithms of *sec.* l, *cosec.* p, *cos.* S, and *sin.* $(S-a)$: then one-half their sum is the sine of half the hour angle in degrees, &c. This must be doubled and converted into time for the hour angle.

(*e*) If the hour angle be *east*, that is, if the time at place be before noon, subtract it from 24 hours, the remainder is the apparent time of the day before; if the hour angle be *west*, or in the afternoon, the hour angle is the apparent time at place of the given day.

(*f*) Reduce this to mean time by applying the equation of time as directed at the top of page 1.

(*g*) The difference between the mean time at ship and Greenwich mean time is the longitude: west if the

time at place be earlier than that at Greenwich, but east if later.

Ex. 270. August 8th, 1887, p.m. at ship in latitude 56° 48′ N., when a chronometer showed August 8th, 4h. 29m. 33·8s., the observed altitude of the sun's lower limb was 30° 57′ 10″, the index correction was − 2′ 10″, and height of eye above the sea 20 feet. Required the longitude, the chronometer having been found fast of Greenwich mean time on 18th July, 7m. 46·5s., and losing daily 2·7s.

For Greenwich time.				*Accumulated error.*
Chronometer showed Aug. 8	4h.	29m.	33·8s.	Rate 2·7s.
Fast 18 July	−	7	46·5	No. days 21·18
	4	21	47·3	14826
Loss in 21·18 days		+	57·2	4236
Greenwich time, Aug. 8	4	22	44·5	57·186

For declination.	*Variation.*	*Eq. of time.*	*Variation.*
Aug. 8 = 16° 10′ 32·7″ N.	For 1hr. 42·56″	+ 5m. 26·41s.	− ·320s.
Correction — 3 6·4	No. hrs. 4·38	Cor. − 1·40	4·38
16 7 26·3 N.	34048	+ 5 25·01	8760
90	12768		1314
	17024		
P. D. = 73 52 34	186·4128		1·40160
	3′ 6·4″		

Observed alt. ☉	30° 57′ 10″	
Index error	− 2 10	
	30 55 0	
Dip	− 4 24	
	30 50 36	
Semidiameter	+ 15 48·8	
	31 6 24·8	
Refraction	1 35·5	
	31 4 49·3	
Parallax	+ 7·6	
True alt.	31 4 57	

(*a*)	31° 4′ 57″		
(*l*)	56 48 0	sec.	10·261566
(*p*)	73 52 34	cosec.	10·017428
	2)161 45 31		
S =	80 52 46	cos.	9·200063
$S-a$ =	49 47 48	sin.	9·882956
			2)19·362013
$\frac{h}{2}$	28° 40′ 6·4″	sin.	9·681006
	2		
Hour angle	57 20 12·8		
	4		
	60)229 20 51·2		

App. time at ship, Aug. 8	3h.	49m.	20·85s.
Equation of time		+ 5	25·01
Mean time at ship, Aug. 8	3	54	45·86
Mean time at Greenwich, Aug. 8	4	22	44·5
Longitude in time		27	58·64
Longitude	6°	59′	39·6″ W.

Ex. 271. 1887, September 1st, at about 9h. 14m. a.m. mean time at ship in latitude by account 45° 51′ 9″ S. longitude 27° E., a chronometer showed 7h. 19m. 43s., and had been found to be fast 3m. 13s. on 3rd May, and slow 12s. on 27th June. If the altitude of the sun's lower limb was 24° 14′ 40″, height of eye above the sea 29 feet, and index error — 3′15″, find the longitude.

For Greenwich time nearly.

Mean time at ship, Aug.	31d.	21h.	14m.
Longitude in time		1	48
M. T. Greenwich, Aug.	31	19	26

For chronometer's rate.

3rd May, fast	3m.	13s.
27th June, slow		12
Loss in 55 days	3	25

Daily rate 2·82s. losing.

For correct Greenwich time.

Chronometer showed, Aug.	31d.	19h.	19m.	43s.
Slow, 27th June			+	12
Aug.	31	19	19	55
Loss in 65·8 days		+	3	5·6
Greenwich time, Aug.	31	19	23	0·6

For accumulated error.

Rate	2·82s.
No. days	65·8
	60)185·556
	3m. 5·6s.

For declination.

Sept. 1 =	8°	18′	47·5″	N.
Correct.	+ 4	11·4		
True dec.	8	22	59	N.
S. P. D.	98	22	59	

Variation.

For 1h. + 54·41″
4·62
60)251·3742
4′ 11·4″

Eq. of time.

+ 0m. 3·85s.
— 3·63
+ 0 0·22

Variation.

— ·785
4·62
3·62670

Observed alt. $\odot$	24°	14′	40″
Index error	—	3	15
	24	11	25
Dip	—	5	18
	24	6	7
Semidiameter	+	15	53·5
	24	22	0·5
Refraction	—	2	7·4
	24	19	53·1
Parallax		+	8
True alt.	24	20	1

(*a*)	24°	20′	1″	
(*l*)	45	51	9	sec. 10·157076
(*p*)	98	22	59	cosec. 10·004666
	2)168	34	9	
s	84	17	4·5	cos. 8·998204
s—*a*	59	57	3 5	sin. 9·937316
				2)19·097262
$\frac{h}{2}$	20°	42′	48·8″	sin. 9·548631
	2			
Hour angle	41	25	37·6	
			4	
	165	42	30·4	

Eastern hour angle	2h.	45m.	42·51s.	
	24			
Apparent time at ship, Aug.	31d.	21h.	14m.	17·49s.
Equation of time				+ 0·22
Mean time at ship, Aug.	31	21	14	17·71
Mean time at Greenwich, Aug.	31	19	23	0·6
Longitude in time		1	51	17·11
		60		
		4)111	17·11	
Longitude		27°	49′ 17″ E.	

SECOND METHOD for finding longitude by chronometer. (pag 145, IA)

RULE (*a*) Find the Greenwich date from the chronometer.

(*b*) Correct the declination, equation of time, and altitude, and find the zenith distance.

(*c*) Under the latitude put the declination, then if both be north or both south, take their difference; if one be north and the other south, take their sum.

(*d*) Find half the sum and half the difference of the result found in (*c*) and the zenith distance.

(*e*) Add together log. secant of latitude, log. secant of declination, log. sine of half sum, and log. sine of half difference found in (*d*). Half this sum is log. sine of half the hour angle.

(*f*) Finish as in the first method.

Ex. 272. 1887, September 1st, at about 9h. 14m. a.m. mean time at ship in latitude 45° 51′ 9″ S., longitude by account 27° E., a chronometer showed 7h. 19m. 43s., and had been found to be fast 3m. 13s. on 3rd May, and slow 12s. on 27th June. If the altitude of the sun's L.L. was 24° 14′ 40″, height of eye 29 feet, and index error — 3′ 15″, find the longitude.

The student will notice this is the same question as the one last worked, and hence we may take the corrected data as there found—

Latitude	45° 51′ 9″ S.	sec.	10·157074
Declination	8 22 59 N.	sec.	10·004666
A	54 14 8		
Zen. dist. B	65 39 59		
$\frac{A+B}{2}$	59 57 3·5	sin.	9·937316
$\frac{A \infty B}{2}$	5 42 55·5	sin.	8·998205
			2)19·097261
$\frac{h}{2}$ =	20° 42′ 48·8″	sin.	9·548631

This question must now be finished as in the last example; and as the hour angles are equal the resulting longitude will be the same.

LONGITUDE BY STAR, MOON, OR PLANET.—The method to be followed is similar to that for the sun; except that the data

for the given object must be taken from the "Nautical Almanac" instead of that for the sun. After the hour angle is found the following formula must be used:—

Mean time at place = hour angle of the object + R. A. of object — R. A. of the mean sun.

Rule:—(*a*) Proceed as in the case of the sun until the hour angle is found. If the altitude be increasing subtract the hour angle in time from 24 hours.

(*b*) Find the corrected right ascension of the mean sun and of the other object.

(*c*) To the western hour angle of the object add its right ascension and subtract the right ascension of the mean sun. This gives mean time at place.

(*d*) Finish as before.

Ex. 273. 1887, October 11, at 12h. 40m. p.m. mean time at ship in latitude 26° 30′ 20″ S., longitude by account 20° W., the chronometer showed 4h. 5m. 14s., the altitude of Aldebaran east of the meridian was 34° 22′ 30″; index error + 5′ 13″; height of eye 17 feet. Required the longitude.

The chronometer was 2h. 5m. 4s. fast of Greenwich mean time on 13th September, and losing daily 2·1s. In reading the sextant it was afterwards found that the observer had read 34° 22′ 30″ instead of 34° 32′ 30″ for the altitude of the star. What was the correct longitude?

For Greenwich time nearly.

Mean time ship, Oct.	11d.	12h.	40m.	0s.
Longitude, W.	+	1	20	0
G. M. T. nearly, Oct.	11	14	0	0

Longitude in time.

Long. 20° 0′ W.
4

60)80 0

1h. 20m. 0s.

For correct G. M. T.

Chronometer showed	4h.	5m.	14s.
Fast 13 Sept.	— 2	5	4
	2	0	10
Loss in 28·58 days	+	1	0
	2	1	10

∴ G. M. T., October 11d. 14h. 1m. 10s.

Accumulated error.

28·58 days
2·1 rate

2858
5716

60·018

For sidereal time.

R. A. of mean sun	**13h.**	**19m.**	**1·95s.**
Acceleration for 14h.		**2**	**17·99**
Acceleration for 1m.			**0·16**
Acceleration for 10s.			**0·03**
R. A. mean sun	**13**	**21**	**20·13**

To correct star's alt.

Observed alt.	**34°**	**22′**	**30″**
Index error	**+**	**5**	**13**
	34	**27**	**43**
Dip	**—**	**4**	**4**
	34	**23**	**39**
Refraction	**—**	**1**	**23**
True alt.	**34**	**22**	**16**

Data for star Aldebaran.

R. A. 4h.29m.28·9s.. Dec. 16°16′54·3″. S. P. D. 106°16′54·3″

To find the longitude.

(*a*)	34°	22′	16″	
(*l*)	26	30	20	sec. ·048230
(*p*)	106	16	54	cosec. ·017776
	2)167	9	30	
(*S*)	83	34	45	cos. 9·048558
(*S* — *a*)	49	12	29	sin. 9·879146
				2)18·993710
$\frac{h}{2}$	18°	17′	49·8″	sin. 9·496855
Hour angle *	36	35	39·6	
			4	
	60)146	22	38·4	
Eastern hour angle	2h.	26m.	22·64s.	
	24			
Western hour angle *	21	33	37·36	
Right ascension *	4	29	28·90	
R. A. meridian	26	3	6·26	
R. A. mean sun	13	21	20·13	
M. T. ship, Oct. 11	12	41	46·13	
G. M. T., Oct. 11	14	1	10	
Longitude in time	1	19	23·87	
	60			
	4)79	23·87		
Longitude	19°	50′	58″ W.	

To find the hour angle by the second method.

Lat.	26°	30′	20″	S.	sec.	·048230
Dec.	16	16	54	N.	sec.	·017776
A	42	47	14			
Zen. dist. B	55	37	44			
$\frac{A+B}{2}$	49	12	29		sin.	9·879146
$\frac{A \backsim B}{2}$	6	25	15		sin.	9·048558
						2)18·993710
$\frac{h}{2}$	18	17	49·8		sin.	9·496855

Having found $\frac{h}{2}$ the solution is completed as before.

To find the error in the longitude due to 10′ error in altitude.

On pages 147 or 149 the following formula was proved :—

$$dh = dz \frac{\sin. z}{\sin. p \,.\, \sin. l' \,.\, \sin. h}; \text{ that is,}$$

error in the hour angle = error in alt. × cos. alt. × sec. dec. × sec. lat. × cosec. h.

Error in alt.	10′ = 600″			log.	2·778151
alt.	34°	22′	16″	cos.	9·916664
dec.	16	16	54	sec.	·017776
lat.	26	30	20	sec.	·048230
h	36	35	40	cosec.	·224646
Error in hour angle	967·1″			log.	2·985467

Because the altitude was assumed too small, therefore the eastern hour angle obtained is too great by 967·1″ or 16′ 7″, and the western hour angle too small by that amount; and hence the longitude obtained, viz. 19° 50′ 58″ W. is too great by 16′ 7″, and therefore the true longitude is 19° 34′ 51″ W.

When the student can calculate the azimuth of an object he will find the true bearing of Aldebaran was N. 43° 53′ 27″ E., and then the following formula, proved on pages 147 or 149, will be much shorter than the one now used :—

$$dh = \frac{dz}{\sin. l' . \sin. Z'}, \text{ or}$$

error in the hour angle = error in alt. × sec. lat. × cosec. az.

Error in alt.		600″		log.	2·778151
lat.	26°	30′	20	sec.	·048230
az.	43	53	27	cosec.	·159087
Error in hour angle		967·1		log.	2·985468

The results agreeing exactly with that obtained by the longer formula.

Exercise X.

Ex. 274. 1887, July 1st, a.m. at ship in latitude 62° 8′ 45″ N., the observed altitude of the sun's L. L. was 25° 0′ 40″. Index correction — 30″; height of the eye 27 feet; time by chronometer 1d. 6h. 0m. 1s., which on 17th May had been found to be fast of Greenwich mean time 2m. 4s., and gaining daily ·97s. Find the longitude.

Ex. 275. 1887, September 1st, at about 4h. 5m. p.m. mean time at ship in latitude 2° 10′ N., longitude by account 55° 40′ E., the chronometer showed 10h. 57m. 5s., the observed altitude of the sun's upper limb was 29° 11′ 40″. Index correction for the sextant — 50″; height of eye above the sea 22 feet. Required the longitude.

On 24th March the chronometer was slow of G. M. T. 56m. 34s., and on 4th May the chronometer was slow of G. M. T. 1h. 3m. 27s.

Ex. 276. 1887, April 15th, a.m. at ship in latitude 29° 46′ S. the chronometer showed noon of the same day. It had been found on 8th January to be fast 2h. 52m. 22s., and on 3rd February to be fast 2h. 56m. 50s. The observed altitude of the sun's L. L. was 29° 27′ 50″; index error + 2′ 41″; height of the eye 9 feet. Find the longitude.

If after finding the longitude there had been found to be an error of 6′ too great in latitude, what was the true longitude?

Ex. 277. 1887, October 31st, at about 3h. 25m. p.m. mean time at ship, when a chronometer showed 3h. 19m. 52s., the observed altitude of the sun's U. L. was 23° 46′; height of the eye above the sea 25 feet; index correction for the sextant — 9′ 59″; latitude by account 24° 17′ N.; longitude by account 180° W. Find the longitude. On 23rd July the chronometer was 7m. 44s. slow of G. M. T., and gaining daily ·5s. Required the longitude.

If after working there had been discovered an error of 10′ too

little in reading the sextant, what is the corresponding error in the longitude?

Ex. 278. 1887, in latitude 31° 22′ 45″ N. the observed altitude of the sun's U. L. was 26° 0′ 15″, height of the eye 27 feet, and index error — 11′, when the chronometer showed 11d. 11h. 29m. 36s., and was slow of G. M. T. 59m. on 7th April, and gaining daily ·85s. If the ship time be a.m. July 12th, find the longitude.

Ex. 279. 1887, August 5th, at about 2h. 30m. p.m. mean time at ship in latitude 30° 27′ 45″ S., the observed altitude of the sun's U. L. was 31° 32′; index correction for sextant — 4′ 44″; height of eye above the sea 19 feet. The chronometer showed 8h. 19m. 41s. a.m. of the same day, and had been found to be fast of G. M. T. on 23rd May 7m. 33s., and losing daily $2\frac{1}{4}$s. Find the longitude.

Ex. 280. 1887, February 18th, at about 6h. 17m. p.m. mean time at ship, the altitude of α Piscis Australis W. of the meridian was 28° 16′ 10″, height of the eye 18 feet, index error — 25″, in latitude 47° 10′ S., and longitude by account 27° 42′ E. At the time a chronometer showed 3h. 25m. 22s., which was slow of G. M. T. on February 2nd 1h. 0m. 20s., losing daily 1s. Required the longitude.

Ex. 281. 1887. On January 3rd, at about 6h. 30m. a.m. mean time at ship in latitude 51° 16′ N., longitude by account 143° 30′ W., when a chronometer showed 4h. 13m. 27s., the observed altitude of Pollux W. of the meridian was 23° 54′ 40″; index correction + 5′ 38″; height of eye above the sea 14 feet. The chronometer was 5m. 20·5s. fast of G. M. T. at noon on 15th December, 1886, and its daily rate was 4·8s. gaining. Find the longitude.

If after working it had been discovered that the true reading should have been 23° 49′ 40″, what was the true longitude?

Ex. 282. 1887, February 11th a.m. at ship, a chronometer showed 9h. 9m. 4s. It was correct for G. M. T. on 29th September, 1886, and slow of G. M. T. 1m. on 12th December, 1886. The observed altitude of the sun's U. L. was 14° 28′; height of eye 27 feet; index error + 2′ 33″. If the latitude was 50° 22′ N., and by account the ship was on the meridian of Greenwich, find the longitude.

Ex. 283. 1887, September 1st, at about 7h. 15m. a.m. mean time at ship, which was on the equator, in longitude by account 60° W., the observed altitude of the sun's L. L. was 18° 3′,

height of eye above the sea 32 feet, and index error of the sextant $-3' 27''$. If the chronometer showed 10h. 57m. 27s., and was correct on 30th June, and slow 4m. 50s. on 27th July, what was the longitude?

Ex. 284. 1887, September 22nd p.m. at ship in latitude 9° 7′ 9″ N., the altitude of the sun's L. L. was 53° 29′ 15″; index error $-3' 14''$; height of eye above the sea 21 feet. The time by chronometer was, September 22nd, 13h. 47m. 27s., which had been found to be slow 1m. 58s. on 18th August, and gaining daily ·87s. Required the longitude.

If an error of 4′ too great had been made in the altitude, find the corresponding error in longitude.

Ex. 285. 1887, August 20th, at about 8h. 20m. a.m. mean time at ship in latitude 12° 15′ 5″ S., longitude by account 168° 40′ W., when a chronometer showed 7h. 19m. 19s., the altitude of the sun's L. L. was 29° 46′ 15″, the index correction for the sextant was $-2' 17''$, and the height of the eye above the sea was 23 feet. If on June 8th the chronometer was slow 18m. $9\frac{1}{2}$s., and gaining daily ·9s., what was the longitude?

Ex. 286. 1887, November 21st, at about 9h. p.m. mean time at ship in latitude 41° 14′ 20″ N., longitude by account 41° 39′ W., a chronometer showed 0h. 57m. 54s., which at noon on November 15th was fast 1h. 10m. 11s, and losing 4·5s. daily; the observed altitude of Pollux east of the meridian was 12° 32′ 30″; index error $-1'$; height of the eye 17 feet. Required the longitude.

If an error in the altitude had been detected after finding the longitude, amounting to 10′ too little, what was the corresponding error in the hour angle?

Ex. 287. 1887, November 29th, at about 1h. 40m. a.m. mean time at ship in latitude 42° 26′ S., longitude by account 57° W.; the observed altitude of the moon's U. L. west of the meridian was 21° 45′ 10″; index correction for the sextant $-2' 13''$; height of eye above the horizon 22 feet. The time by chronometer was 4h. 0m. 52s., and it had been found to be fast for G. M. T. on 28th July 1m. 13s., and on 5th September slow 39s. Required the longitude.

If the altitude had been discovered 15′ too little, what was the true longitude?

Ex. 288. Explain clearly what you understand by time, and state how and with what units it is reckoned. The "Nautical Almanac," at page 1, gives certain elements "at apparent noon," and, at page 2, "at mean noon." Give definitions of these

two terms: which page do you use in finding the sun's declination in the two problems "latitude by meridian altitude" and "longitude by chronometer"? Give your reasons. *E.* 1878.

Ex. 289. Define apparent solar time, mean solar time, and equation of time. What is the time shown by a chronometer? What corrections must be applied to the indications of a chronometer when it is used at sea? For what problem is this used? *E.* 1883.

Ex. 290. By drawing a spherical triangle show how the apparent time is obtained in working longitude by chronometer. What parts of this triangle have you given, and what do you wish to find? Find them. *E.* 1878.

Ex. 291. What is meant by a Greenwich date? Show how it may be found; and explain its use in taking out from the "Nautical Almanac" the sun's declination and the equation of time at any time for any place.

In finding the longitude from an observed altitude of the sun how do you find the mean time from the sun's hour angle? In this case do you take out the equation of time from page 1 or page 2 of the "Nautical Almanac," and why? *E.* 1881.

Ex. 292. Define sidereal time, apparent solar time, and mean solar time, illustrating your definitions by projections, (1) on the equinoctial, (2) on the horizon. In a "sun chronometer" what angle do you deduce from your observation combined with other data, and what must you apply to this before you can compare it with the time indicated by your chronometer? *E.* 1869.

Ex. 293. At a place whose latitude is known I observe the altitude of a heavenly body, the declination of which I find in the "Nautical Almanac;" show how I can determine the hour angle. *A.* 1874.

Ex. 294. Explain by means of a figure the rule for finding the longitude by chronometer. *Royal Naval College,* 1863.

Ex. 295. Describe a method applicable at sea for determining the longitude and latitude of a place on the earth.
Second B. A. and B.Sc. London, 1875.

Ex. 296. Explain the method of determining the longitude by chronometer, showing how the work differs in the two cases, (1) when the body observed is the sun, (2) when the body observed is a star. Prove the formula required for this problem. *A.* 1883.

Ex. 297. If h be the hour angle of a heavenly body, d its

declination, z its zenith distance, and l the latitude of the observer, prove—

$$\sin.^2 \frac{h}{2} = \frac{\sin. \frac{1}{2}\{z + (l \underset{\backsim}{+} d)\} . \sin. \frac{1}{2}\{z - (l \underset{\backsim}{+} d)\}}{\cos. d . \cos. l}.$$

Distinguish between the two cases indicated by $l \underset{\backsim}{+} d$. Explain fully how to deduce from this formula the mean time when the body observed is a fixed star, and hence the longitude of the place. *A.* 1869.

Ex. 298. Describe some method by which you would find practically the difference of longitude of two places.

Second B.A. and B.Sc. London, 1867.

Ex. 299. At a place in latitude 36° N., and sun's declination 15° N., my shadow was to my height as 9 to 7. Find the sun's hour angle. *Royal Naval College*, 1867.

Ex. 300. The error in the hour angle is the least for a given error in the altitude when the heavenly body is in the prime vertical. Required a proof. *For Beaufort Testimonial*, 1865.

Ex. 301. At a place of known latitude, the altitude of a star is observed, whose declination and right ascension are likewise known. Determine the time of observation.

Second B.A. and B.Sc. London, 1875.

Ex. 302. The expression for finding the error in the hour angle corresponding to the error in altitude can be found by the following formula:—

Error in hr. $\angle = \frac{1}{15}$ *sec. lat.* $\times$ *cosec. azi.* $\times$ *error in alt.*

Prove this expression. *Example.* Find the error in the hour angle corresponding to an error of five minutes in the altitude in latitude 50° N., when the bearing of the observed body is S. 60° E. *Royal Naval College*, 1865.

Ex. 303. In a star chronometer the hour angle of a star was found to be 15h. 21m. 30s. The R. A. of the mean sun at the preceding Greenwich mean noon was 10h. 10m. 7s. The R. A. of the star was 7h. 12m. 8s. Draw a figure on the plane of the equator, and find ship's mean time, longitude being 40° E.

For Lieutenant, 1873.

Ex. 304. During the night of 11th March, 1887, *Capella* was observed at London, latitude 51° 32′ N., to be 20° 30′ above the horizon. At what time was the observation made? *A.* 1869.

Ex. 305. At a place whose latitude is 45° 20′ N., find the error in the hour angle corresponding to an error of 4′ in the altitude when the bearing of the body is S.E. *H.* 1874.

Ex. 306. On May 22nd, 1887, the altitude of *Capella* is observed, and its hour angle calculated and found to be 7h. 50m. 30s.: the chronometer time corrected for error and rate is 9h. 30m. 40s. Find the longitude, and draw a figure illustrating this example. *A*, 1879.

Ex. 307. What is the best time for observing an altitude for longitude, and why? Investigate the formulæ which prove this. Compare the errors in the apparent times caused by an error of 5 minutes in an observed altitude when the bearing is E. ½ S, and S. ½ E. respectively. *H*. 1881.

CHAPTER XII.

Error and rate of time-keeper obtained—*First,* by finding the hour angle of an object—*Secondly,* by equal altitudes of the same star on the same side of the meridian on different days—Examples—*Thirdly,* by equal altitudes of the same star on different sides of the meridian on the same day—Example—*Fourthly,* by calculating the equation of equal altitudes—Proof of formula—On making the observation—To find the time when an object has the same altitude west as it had east of the meridian—Rules for the equation of equal altitudes—Example—Exercise—Examination.

The time-keeper for sea purposes is the chronometer, and as its use is to find Greenwich mean time from what it indicates, its *error* is therefore what it differs from the mean time at Greenwich, *fast* when it shows a later time, and *slow* when it shows an earlier time than the true Greenwich mean time. The *rate* of a chronometer is its daily loss or gain; and it is said to be *gaining* when it goes too fast, and to be *losing* when it goes too slow.

Although the principles on which the chronometer is constructed are so rigidly correct, still it is impossible to produce a perfect instrument, and a change in the rate of a chronometer is often found. This necessitates a method or methods for detecting the change. At a stationary observatory a perfectly regulated sidereal clock or a transit instrument gives the most exact method for rating a chronometer; but as these cannot be used by mariners, who are necessarily constantly changing their positions, recourse to other methods must be had. There are four methods, which are easy of application by sailors, for finding local time, and from that Greenwich time is deduced.

First method, *by finding the hour angle of a body at a place whose latitude and longitude are known.*—This is done by taking the altitude of the object when near the prime vertical, for

reasons shown in the last chapter: and because the sea horizon is so uncertain, an artificial horizon should be used. Having obtained the hour angle the mean time at place is found, and by the application of the longitude in time that at Greenwich is deduced. What the chronometer differs from this is its error. This process should be repeated at some future day under as nearly as possible the same conditions; that is, the observations should be made by the same observer with the same instrument, on the same object at the (or nearly so) same altitude on the same side of the meridian, and its error again found. The rate is then calculated by dividing the change in error by the number of days elapsed between the observations.

Ex. 308. On 23rd March, at 8h. 30m. a.m., the chronometer was found by observation to be slow of G. M. T. 5m. 13s. On 18th June at 4h. 30m. p.m. it was found slow 2m. 49s. What is its daily rate?

March 23,	8h.	30m.	a.m. slow	5m.	13s.
June 18,	4	30	p.m. slow	2	49
			Gain in $87\frac{1}{3}$ days $=$	2	24
			$=$		144

$$\therefore \text{ Daily rate } = \frac{144}{87\frac{1}{3}}$$

$$= 1{\cdot}65\text{s. gaining.}$$

SECOND METHOD, *by equal altitudes of the same star on the same side of the meridian on different days.*—We have already shown, p. 36, that the mean solar day is longer than the sidereal day by 3m. 55·91s. of mean solar time; that is, the same star returns to the meridian 3m. 55·91s. earlier each day than on the preceding one, and will, therefore, at the same place, return to any one altitude on the same side of the meridian earlier each day by that interval. Hence the difference between 3m. 55·91s. and the interval shown by a chronometer when a star has equal altitudes on the same side of the meridian on two consecutive days is the rate. In practice it is preferable to allow several days to elapse between the observations, and to select a star whose altitude is as great as possible and is changing quickly, and should therefore lie near the prime vertical. *If the interval by the chronometer be less than 3m. 55·91s. per day, it is gaining; but if*

the interval by chronometer be greater than that, it is losing.

Ex. 309. On August 6th, at 10h. 7m. 46s., the altitude of Arcturus by an artificial horizon west of the meridian was 46° 31′. On August 29th, at 8h. 34m. 50·5s. by the same chronometer, the star had the same altitude west with the same sextant. Required the rate of the chronometer.

		h.	m.	s.
August 6, chronometer showed		10h.	7m.	46s.
August 29, " "		8	34	50·5
Diff. in time for eq. alts. in 23 days	=	1	32	55·5
" " " 1 day	=		4	2·41
but true " " " "	=		3	55·91
Hence rate	=			6·5 losing.

If the error be required it may be calculated as by the first method.

Third method, *by equal altitudes of the same star on different sides of the meridian on the same day.*—Because a star does not change its declination, it will, at the same place, have equal hour angles east and west of the meridian when it has equal altitudes; and hence, midway between the times when it had equal altitudes on the same day, it must have been on the meridian. The method to get the error and rate of the chronometer is as follows:—

(*a*) Let the times by chronometer be noted when the same star had equal altitudes on the same day E. and W. of the meridian. Take the mean of these times; this is the time by chronometer when the star was on the meridian.

(*b*) Calculate the time when the star was on the meridian, p. 96.

(*c*) Take the difference between the time observed by the chronometer of the star's transit and that by calculation: this is the error of the chronometer for mean time at place. The rate is obtained by repeating the observations on a subsequent day.

Ex. 310. 1877, Feb. 23. The following equal altitudes of Regulus were taken at the Navigation School, Plymouth, lat. 50° 22′ 25″ N., long. 4° 7′ 16·5″ W., to find the error of the chronometer for Greenwich mean time.

Star E. of the meridian.				*Star W. of the meridian.*		
Chro. times	8h.	27m.	0s.	15h.	29m.	13s.
	8	29	10	15	27	3
	8	30	50	15	25	23
	8	32	32	15	23	41
	8	34	38	15	21	35
	5)42	34	10	5)77	6	55
				15	25	23
Time E. mer.	8	30	50			
Time W. „	15	25	23			
	2)23	56	13			
	11	58	65 =time by chro. of star's transit.			

To find Greenwich time nearly of star's transit.

R. A. of Regulus on Feb. 23	10h.	2m.	22·32s.
R. A. of mean sun at noon	22	12	14·50
Date at place nearly of star's transit	11	50	7·82
Long. 4° 7′ 16·5″ W. in time +		16	29·1
Greenwich time nearly of star's transit	12	6	36·92

For the mean sun's right ascension.

Sidereal time mean noon at Greenwich	22h.	12m.	14·50s.
Acceleration for { 12h.		1	58·28
6m.			·99
37s.			·10
Mean sun's R. A. for Greenwich date	22	14	13·87

For error of chronometer.

R. A. of Regulus on Feb. 23	10h.	2m.	22·32s.
R. A. mean sun at Greenwich date	22	14	13·87
Correct date at place of star's transit	11	48	8·45
But chronometer showed	11	58	6·5
∴ Chronometer fast for M. T. at place		9	58·05
Greenwich time later for 4° 7′ 16·5″ W.		16	29·1
∴ Chronometer slow for G. M. T.		6	31·05

FOURTH METHOD *by calculating the equation of equal altitudes.*—Because the sun changes his declination in the interval between his having the same hour angle when east and when west of the meridian, the mid time between the observations will not coincide with the time he was on the meridian; but will be later if the western hour angle be the greater, and earlier if the eastern hour angle be the greater. Hence a correction must be applied to the mid-time shown by the chronometer to obtain the time of transit or apparent noon. *This correction is called the equation of equal altitudes.*

Now the hour angle of a body is obtained from the formula—

$$\text{Cos. } h = \frac{\cos. z - \cos. p \,.\, \cos. l'}{\sin. p \,.\, \sin. l'},$$

where in the case of the sun p lies between $66\frac{1}{2}°$ and $113\frac{1}{2}°$; but between these limits cos. p decreases faster than sin. p increases; and, therefore, when p is increasing the value of the numerator increases faster than the denominator and the fraction becomes larger; that is, cos. h increases when p increases, hence the hour angle becomes smaller as the polar distance becomes greater, and *vice versâ*. And, therefore, for equal altitudes of the sun at the same place his eastern hour angle will be greater than his western one when his polar distance is increasing; and then the mid time of the observations must be before the time of transit; hence, *if the polar distance be increasing the equation of equal altitudes must be added to the mid time shown by the chronometer.* On the contrary, the western hour angle will be the greater of the two under the same conditions when the polar distance is decreasing, and the mid time must then be after the time of transit; and, therefore, *the equation of equal altitudes must be subtracted from the mid time by the chronometer when the polar distance is decreasing.*

"Let h and h' represent the eastern and western hour angles (reckoned in time) when the sun had equal altitudes upon a certain day, then $(24 - h)$ hrs. and $(24 + h')$ hrs. will also represent the apparent times of observation, and their mean $\left(24 + \frac{h' - h}{2}\right)$ hrs. will exceed the true time of noon by $\frac{h' - h}{2}$, or by half the difference of the hour angles, provided h' be greater than h, or the western hour angle be the greater. But the expression for the mean may also be written thus:

$\left(24 - \frac{h-h'}{2}\right)$ hrs. if the eastern hour angle be the greater; and in this case the middle time is before the time of noon by half the difference of the hour angles, or $\frac{h-h'}{2}$.

Half the difference of the hour angles is called the equation of equal altitudes,"[1] and from what has been said, it is plain its amount and sign will depend on the change in the hour angle caused by the change in the polar distance between the times when the sun had equal altitudes. Its amount is found as follows:—

Beginning with the fundamental formula—

$$\begin{aligned} \text{Cos. } z &= \cos. p \,.\, \cos. l' + \sin. p \,.\, \sin. l' \,.\, \cos. h \\ &= \cos. p \,.\, \sin. l + \sin. p \,.\, \cos. l \,.\, \cos. h\,; \end{aligned}$$

and considering h and p as the variables, we have by differentiating

$$\begin{aligned} o = &- \sin. p \,.\, dp \,.\, \sin. l + \cos. p \,.\, dp \,.\, \cos. l \,.\, \cos. h \\ &\qquad - \sin. p \,.\, \cos. l \,.\, \sin. h \,.\, dh\,; \end{aligned}$$

$$\begin{aligned} \therefore\ dh &= \frac{\cos. p \,.\, \cos. l \,.\, \cos. h - \sin. p \,.\, \sin. l}{\sin. p \,.\, \cos. l \,.\, \sin. h} \,.\, dp\,. \\ &= dp\,(\cot. p \,.\, \cot. h - \tan. l \ \text{cosec. } h) \quad .\ .\ \text{I.} \\ &= dp\,(\tan. \text{dec. } \cot. h - \tan. \text{lat. cosec. } h). \end{aligned}$$

We have already shown that the change in the hour angle will be of a contrary kind to the change in the polar distance, or, if the polar distance be increasing, then the western hour angle will be smaller, and *vice versâ;* this amounts to making dh negative; then

$$\begin{aligned} - dh &= dp\,(\tan. \text{dec. } \cot. h - \tan. \text{lat. cosec. } h) \\ \text{or } \frac{dh}{2} &= \frac{dp}{2}\,(\tan. \text{lat. cosec. } h - \tan. \text{dec. } \cot. h) \ .\ (a) \end{aligned}$$

Now dp is the small change in the polar distance during the interval, and dh the resulting small change in the hour angle; and as the equation of equal altitudes is half the difference in the hour angles, let $\frac{dp}{2}$, or half the change in the polar distance during the interval $= c$, then

Equa. of eq. alt. $= c\,(\tan. \text{lat. cosec. } h - \tan. \text{dec. } \cot. h)$ II.

when h is very nearly equal to half the elapsed time.

[1] Riddle's "Navigation and Nautical Astronomy."

Separating the two times in equation II.—

$$A = c \text{ tan. lat. cosec. } h,$$
$$B = c \text{ tan. dec. cot. } h;$$

then equa. of eq. alt. $= A \pm B$.

In equation I., if p be greater than 90°, cot. p becomes negative, and both terms of the equation will then be of the same sign; but p will be greater than 90° when the latitude and declination have different names: hence the rule, *if the latitude and declination have contrary names add A and B together; but if they be of the same name take the difference between A and B for the equation of equal altitudes.*

Because cosec. h is always greater than cot. h, therefore the first term always exceeds the second when the latitude equals or exceeds the declination; hence in the temperate zones A is always greater than B, but within the tropics B may in some cases be greater than A; when this is so the rule for the application of the equation of equal altitudes must be reversed, and then the equation of equal altitudes is additive when the polar distance is decreasing, and subtractive when the polar distance is increasing.

The formula II. may be obtained without the use of the differential calculus; and for those who have not read that branch of mathematics we add the following proof:—

$$\begin{aligned}\text{Cos. } z &= \cos. p \,.\, \cos. l' + \sin. p \,.\, \sin. l' \,.\, \cos. h \\ &= \cos. p \,.\, \sin. l + \sin. p \,.\, \cos. l \,.\, \cos. h \quad . \quad . \quad . \quad (1)\end{aligned}$$

It has been shown, if p take an increment dp, and z and l remain constant, then h will take a decrement dh, and

$$\cos. z = \cos. (p + dp) . \sin. l + \sin. (p + dp) . \cos. l . \cos. (h - dh). \quad (2)$$

Subtracting (2) from (1)—

$$\begin{aligned}o = \sin. l \, \{\cos. p - \cos. (p + dp)\} \\ + \cos. l \, \{\sin. p \,.\, \cos. h - \sin. (p + dp) \cos. (h - dh)\}\end{aligned}$$

Transposing and expanding—

$$\begin{aligned}(\sin. p \,.\, \cos. dp + \cos. p \,.\, \sin. dp) (\cos. h \,.\, \cos. dh + \sin. h \sin. dh) \\ - \sin. p \,.\, \cos. h = \tan. l \,(\cos. p - \cos. p \,.\, \cos. dp + \sin. p \,.\, \sin. dp).\end{aligned}$$

Now as dp and dh are considered very small, their cosines may be taken equal to unity, then

$$\begin{aligned}(\sin. p + \cos. p \,.\, \sin. dp) (\cos. h + \sin. h \,.\, \sin. dh) - \sin. p \,.\, \cos. h \\ = \tan. l \,(\cos. p - \cos. p + \sin. p \,.\, \sin. dp)\end{aligned}$$

Multiplying up—

$$\text{Sin.}\,p\,.\,\cos.\,h + \cos.\,h\,.\,\cos.\,p\,.\,\sin.\,dp + \sin.\,p\,.\,\sin.\,h\,.\,\sin.\,dh + \cos.p\,.\,\sin.dp\,.\,\sin.\,h\,.\,\sin.\,dh - \sin.p\,.\,\cos.h = \tan.l\,.\,\sin.\,p\,.\,\sin.dp.$$

Cos. p and sin. h are each proper fractions, and sin. dp and sin. dh are each very small quantities, therefore no material error will be introduced by neglecting the term containing the product of these four quantities, then

$$\sin.\,p\,.\,\sin.\,h\,.\,\sin.\,dh = \sin.\,dp\,(\tan.\,l\,.\,\sin.\,p - \cos.\,p\,.\,\cos.\,h).$$

Again, because dh and dp are very small, these quantities may be written $dh\,.\,\sin.\,1''$ and $dp\,.\,\sin.\,1''$ respectively,

$$\therefore\ dh\,.\,\sin.\,1'' = dp\,.\,\sin.\,1''\,\frac{\tan.\,l\,.\,\sin.\,p - \cos.\,p\,.\,\cos.\,h}{\sin.\,p\,.\,\sin.\,h},$$

$$\text{or } \frac{dh}{2} = \frac{dp}{2}\,(\tan.\ \text{lat.}\ \text{cosec.}\ h - \tan.\ \text{dec.}\ \cot.\ h),$$

which agrees with the equation marked (a). The proof is completed as before.

In high latitudes, tan. lat. becomes very great, and the results less and less trustworthy as the latitude increases. The best method in such cases is to compute each hour angle separately with the correct polar distance for each altitude, and then deduce the error of the chronometer for each observation. The mean of these errors will very nearly equal the true error at noon.

ON TAKING THE OBSERVATIONS.—In practice it is recommended to take several sets of altitudes, not less than two hours from noon, so that should the weather become cloudy in the interval, one or other of the set of altitudes taken in the morning may be available in the afternoon: and the mean of several observations will probably be more correct than a single one. The vernier of the sextant should be placed at some even number of minutes, as 5, 10, 15, 20, &c., and the observer wait for a contact; then move the vernier to another even five minutes, and so on. It would be preferable to use several sextants, and not move the verniers during the interval, because then the same altitudes absolutely E. and W. of the meridian would be used. The method of equal altitudes taken with an artificial horizon is, perhaps, the most correct method (without the aid of the transit instrument) which can be used for determining the error of the chronometer.

First. Because the index error does not enter into the calculation.

Secondly. The personal error of the observer is also neutralized.

Thirdly. The object being of the same altitude, if there be no alteration in the temperature or humidity of the atmosphere, the refraction (which is the greatest source of error in astronomical calculations) will be the same.

TO FIND THE APPROXIMATE TIME BY CHRONOMETER WHEN AN OBJECT HAD THE SAME ALTITUDE WEST OF THE MERIDIAN AS IT HAD EAST.—To prevent loss of time by preparing for the p.m. observations too soon, or loss of observations by preparing too late, it is necessary to know the time by chronometer nearly when the object will have the same altitude after its transit as it had before. To find this, the error of the chronometer must be known approximately.

Let T be the mean time of transit of the object,

E the estimated error of chronometer for M. T. at place,

then $T + E =$ time by chronometer at transit *if chro. be fast,*

and $T - E =$ " " " *if chro. be slow;*

$\therefore$ $T \pm E =$ time by chronometer at transit.

Let $t =$ time by chronometer of first observation, then $T \pm E - t$ is the interval by chronometer between the time when the first observation was made and the time of transit.

Now if this interval be added to the time of transit by chronometer, that is to $T \pm E$, we shall obtain the time by chronometer nearly after transit when the object has the same altitude as it had before.

$\therefore$ $T \pm E + (T \pm E - t)$ or $2(T \pm E) - t =$ time by chronometer nearly when the p.m. observation should be made for the equal altitude.

$+$ sign to be used before E if the chronometer be fast.

$-$ " " " " " slow.

RULE FOR THE EQUATION OF EQUAL ALTITUDES.—(*a*) Find the middle time by chronometer when the sun had equal altitudes.

(*b*) Find half the elapsed time, and reduce it to degrees, &c. Call this h.

(*c*) Find the Greenwich date for the sun's transit over the meridian by applying the longitude in time to 0h. 0m. 0s.

(*d*) From page I. for the month in the "Nautical Almanac"

take out the declination and equation of time, and correct them for the Greenwich date.

(e) Multiply the hourly difference of the declination by half the elapsed time in hours; call this c.

(f) Then—

$$\log. A = \log. c + \log. \tan. \text{lat.} + \log. \text{cosec.}\, h,$$

and $$\log. B = \log. c + \log. \tan. \text{dec.} + \log. \cot.\, h.$$

(g) A and B must be added together when the latitude and declination are of contrary names; but subtracted one from the other when the latitude and declination are of the same name. The result is the equation of equal altitudes in seconds of arc. This, divided by 15, will give seconds of time.

(h) In the temperate zones, the equation of equal altitudes is to be *added* to the mid time by the chronometer in (a) *if the polar distance be increasing;* but ***sub-tracted*** *if the polar distance be decreasing.* This rule must be reversed if B be greater than A, and latitude and declination be of the same name. The result is the time shown by the chronometer at apparent noon at place.

(i) *For apparent time at place.*—The difference between the last result and noon is the error of the chronometer.

For mean time at place.—Apply the equation of time to noon; this gives the mean time at apparent noon. The difference between this and the result in (h) is the error of the chronometer for mean time at place.

For Greenwich mean time.—Apply the longitude in time to the mean time at apparent noon. This gives Greenwich mean time for apparent noon at place. The difference between this and the result in (h) is the error of the chronometer for Greenwich mean time.

Ex. 311. 1877, July 12. If the following observations be made when the sun had equal altitudes at the Navigation School, Plymouth, latitude 50° 22′ 25″ N., longitude 4° 7′ 16·5″ W., required the error of the chronometer for apparent and mean time at place, and for mean time at Greenwich.

Chronometer ⊙ E. Mer.					*Chronometer ⊙ W. Mer.*		
	21h.	11m.	21s.		2h.	48m.	43s.
	21	12	46		2	47	20
	21	14	48		2	45	19
	21	15	42		2	44	26
	21	16	47		2	43	21
	5)106	11	24		5)13	49	9
	21	14	16·8		2	45	49·8
	2	45	49·8		21	14	16·8
	2)0	0	6·6	Elapsed time =	5	31	33
Mid time by chro.	0	0	3·3	Half elapsed time =	2	45	46·5
				$h =$	41°	26′	37·5″

For Greenwich date at app. noon.

App. noon at place, July 12		0h.	0m.	0s.
Long. in time, W.		+	16	29·1
Greenwich time, July	12	0	16	29·1

Long. in time.

Long.	4°	7′	16·5″ W.
			4
	16	29	6

For sun's declination.		*Variation.*	*Eq. time.*	*Variation.*
July 12th =	21° 59′ 41·4″ N.	1h. = 20·77″	+ 5m. 17·92s.	+ ·319s.
Cor.	− 6·2	·3	+ ·10	·3
True dec.	21 59 35·2	− 6·231.	+ 5 18·02	+ ·0957

The distance of the sun from the elevated pole is increasing.

$c =$ diff. in dec. for 1h. $\times$ half elapsed time
$= 20{\cdot}77'' \times 2{\cdot}763$
$= 57{\cdot}38751.$

$A = c$. tan. lat. cosec. h.

c =	57·39″	log.	1·758836
lat. = 50° 22′ 25″		tan.	·081944
h = 41 26 37·5		cosec.	·179218
A =	104·71″	log.	2·019998
B =	26·25		
	15)78·46		

Eq. eq. alt. = + 5·23 sec.

$B = c$. tan. dec. cot. h.

c =	57·39	log.	1·758836
dec. = 21° 59′ 35″		tan.	9·606258
h = 41 26 37·5		cot.	·054051
B = 26 25		log.	1·419145

$A - B$ is used because the latitude and declination have the same name; and the equation of equal altitudes must be added to the mid time by chronometer because the polar distance is increasing.

For error of chronometer.

Mid time by chronometer	0h.	0m.	3·3s.
Equation of equal altitudes		+	5·23
Time by chronometer at apparent noon	0	0	8·53

(1)	If the chronometer were correct for apparent noon, it would show	0h.	0m.	0s.
	But it shows	0	0	8·53
	∴ Chro. is *fast* for app. time at place			8·53
(2)	Apparent noon at place	0h.	0m.	0s.
	Equation of time	+	5	18·02
	If the chronometer were correct for mean time at place, it would show	0	5	18·02
	But it shows	0	0	8·53
	∴ Chro. is *slow* for mean time at place		5	9·49
(3)	Mean time at apparent noon	0h.	5m.	18·02s.
	Longitude in time	+	16	29·1
	Greenwich mean time at app. noon	0	21	47·12
	Chro. showed at app. noon	0	0	8·53
	∴ Chro. is slow for G. M. T.		21	38·59

EXERCISE XI.

Ex. 312. On June 1st, 1887, at 13h. 17m. 50s., Altair was observed east of the meridian. On July 18th, at 10h. 16m. 29s., Altair had the same altitude and bearing at the same place. Required the rate of the chronometer.

Ex. 313. On October 22nd, 1887, at 10h. 53m. 2s., Pollux was observed east of the meridian. On November 30th, 1887,

at 8h. 16m. 35s., the star at the same place had the same altitude and bearing. Find the rate of the chronometer.

Ex. 314. 1887, November 5th. The following equal altitudes of Rigel were taken at Gibraltar with an artificial horizon—latitude 36° 7′ 18″ N., longitude 5° 21′ 12″ W.—to find the error of the chronometer on Greenwich mean time:—

Star east.			*Star west.*		
Chro. 12h.	26m.	25s.	Chro. 17h.	6m.	19s.
	27	42		5	2
	29	14		3	30

Ex. 315. 1887, January 11th. The following observations were made of Canopus at Cape Town Observatory, in latitude 33° 56′ 3″ S., longitude 18° 28′ 45″ E., when the star had equal altitudes:—

Star east.			*Star west.*		
Chro. 6h.	24m.	18s.	Chro. 11h.	13m.	9s.
	25	40		11	46
	27	0		10	28
	28	22		9	6

Find the error of the chronometer for G. M. T.

Ex. 316. 1887, October 19th. The following times by chronometer were noted when Markab had equal altitudes at Ascension Island; latitude 7° 55′ 30″ S., longitude 14° 25′ 30″ W. Find the error of the chronometer for Greenwich mean time.

Star E. meridian.			*Star W. meridian.*		
Chro. 7h.	33m.	24s.	Chro. 13h.	5m.	20s.
	34	50		3	53
	36	3		2	41

Ex. 317. 1887, December 25th. The following observations of Capella were made at St. Helena to find the error of the chronometer for Greenwich mean time in latitude 15° 55′ S., longitude 5° 44′ W., when the star had equal altitudes:—

Star E. meridian.			*Star W. meridian.*		
Chro. 10h.	6m.	52s.	Chro. 12h.	11m.	0s.
	8	0		9	51

Ex. 318. 1887, May 16th. At Liverpool Observatory, in latitude 53° 24′ 48″ N., longitude 3° W., the equal altitudes of Arcturus were observed at the following times to find the error of the chronometer on Greenwich mean time.

N

Star E. meridian.	*Star W. meridian.*
Chro. 7h. 59m. 13s.	Chro. 13h. 33m. 25s.
8 0 41	31 58
2 30	30 8
3 47	28 52
5 1	27 37

Ex. 319. 1887, November 1st. The following observations were made of Procyon by the same chronometer as in last question at Charleston, South Carolina, latitude 32° 46′ 24″ N., longitude 79° 55′ W. Find the error of the chronometer for G. M. T., and its rate.

Star E. meridian.	*Star W. meridian.*
Chro. 7h. 13m. 40s.	Chro. 13h. 29m. 15s.
15 7	27 48
16 29	26 26

Ex. 320. 1887, July 27th, at Valparaiso, in latitude 33° 1′ 54″ S., longitude 71° 41′ 30″ W., the sun had equal altitudes a.m. and p.m., when the chronometer showed—

Sun E. meridian.	*Sun W. meridian.*
2h. 15m. 25s.	7h. 19m. 50s.
2 17 10	7 21 35

Required its error for apparent time at place, and also for mean time at Greenwich.

Ex. 321. 1887, February 17th, at Melbourne, in latitude 37° 48′ 36″ S., longitude 144° 57′ 42″ E., the sun had equal altitudes a.m. and p.m., when the chronometer showed—

Sun E. meridian.	*Sun W. meridian.*
0h. 10m. 17s.	4h. 29m. 44s.
11 28	28 32
12 39	27 20

Required its error for apparent time at place, and also for Greenwich mean time.

Ex. 322. 1887, November 23rd, at Bermuda, in latitude 32° 19′ N., longitude 64° 52′ W., the sun had equal altitudes a.m. and p.m., when the chronometer showed—

Sun E. meridian.	*Sun W. meridian.*
0h. 47m. 20s.	7h. 51m. 29s.
48 52	49 57
50 21	48 30

Required the error of the chronometer for apparent time at place, and also for mean time at Greenwich.

Ex. 323. 1887, May 31st. Required the error of the chronometer for apparent time at place, and also for G. M. T. at Malta, in latitude 35° 53′ 48″ N., longitude 14° 31′ 12″ E., if the sun had equal altitudes east and west of the meridian, when it showed—

Sun E. meridian.	*Sun W. meridian.*
9h. 12m. 20s.	2h. 47m. 40s.
14 0	46 0
15 30	44 30

Ex. 324. 1887, December 21st, the chronometer showed at Mauritius, in latitude 20° 22′ 36″S., longitude 57° 45′ 42″ E., when the sun was east and west of the meridian—

Sun E. meridian.	*Sun W. meridian.*
3h. 29m. 21s.	0h. 49m. 2s.
30 56	47 27
32 17	46 6

Required the error of the chronometer for apparent time at place, and also for mean time at Greenwich.

Ex. 325. 1887, June 1st. Required the error of a chronometer for apparent time at place, and also for G. M. T. at Madras Observatory, in latitude 13° 4′ 6″ N., longitude 80° 14′ E., if the times shown by it a.m. and p.m., when the sun had equal altitudes, were as follows:—

Sun E. meridian.	*Sun W. meridian.*
8h. 50m. 25s.	3h. 2m. 29s.
8 52 0	3 0 54

Ex. 326. What do you mean by the *rate* of a chronometer? Explain the ordinary method used by navigators for finding the error and rate of a chronometer. On March 7th, 1887, the ship's apparent time at place in longitude 52° 40′ W. was found to be 20h. 25m. 10s., and the chronometer showed 19h. 29m. 40s. What was the error of the chronometer on the mean time at place and on mean time at Greenwich.

On February 20th its error was slow 1h. 7m 0s. Show what it will be on April 10th. *E.* 1869.

Ex. 327. What is the use of the chronometer in navigation? Explain clearly what you mean by the error and rate of the chronometer. There are two chronometers, *A* and *B*; the error of *A* is 1h. 20m. fast, and its rate 2·5s. losing; the error of

B is 1h. 30m. slow, and its rate 5·5s. gaining. What difference of time will they show after 10 days? *E.* 1874.

Ex. 328. What is a chronometer? On 1st January the error of a chronometer on G. M. T. is 2h. 12m. 30s. fast, and its rate 2s. losing; what time would it show at G. M. noon on 31st January? Explain the term "error," "rate," and "G. M. T." What do you mean when you say that 1 hour is equivalent to 15°? *E.* 1882.

Ex. 329. Describe the different methods for finding the error and the rate of a time keeper.

Ex. 330. On June 13th, 1887, at about 9h. 25m. a.m. mean time in latitude 33° S., and longitude 71° 38′ 15″ W., the following set of altitudes of the sun's L. L. was taken by an artificial horizon, index error + 2′; the times were taken by a hack chronometer, which was slow on the standard chronometer 45m. 10s.; the corrected declination for the time of observation was 23° 12′ 37″ N., and the equation of time 0m. 15s. subtractive from the apparent time. Required the error of the chronometer on G. M. T.

Times.	*Altitudes.*	
1h. 16m. 15s.	44° 46′ 30″	
1 16 29	44 50 50	
1 16 45	44 54 50	*E.* 1882.

Ex. 331. April 18th, 1887, at 3h. 21m. p.m. mean time nearly in latitude 24° 36′ S, longitude 26° 18′ W., the observed altitude of the sun's lower limb was 29° 9′ 10″, ind. corr. − 3′ 20″, height of eye 35 feet, when a chronometer showed 4h. 51m. 37s.: also on May 5th, 1887, at 7h. 50m. p.m. mean time nearly in latitude 37° 27′ S., longitude 77° 30′ W., the observed altitude of Sirius (W. of mer.) was 32° 52′ 45″, ind. corr. + 2′ 20″, height of eye 35 feet, when the chronometer showed 1h. 48m. 50s. Required the error of the chronometer on G. M. T. at the second observation, and its daily rate. *A.* 1883.

Ex. 332. What do you mean by "rating a chronometer"? Explain the method of finding the error of a chronometer by equal altitudes of the sun and of a fixed star on the same and on different sides of the meridian. What is the equation of equal altitudes? *A.* 1876.

Ex. 333. Describe fully the different methods in use for rating a chronometer. Investigate the "equation of equal altitudes." *H.* 1878.

Ex. 334. Explain fully the reason for the two signs in the formula—

$$\text{Equa. of eq. alts.} = A \pm B.$$

Ex. 335. Give reasons for the rule: If the polar distance be increasing, the equation of equal altitudes is positive; if the polar distance be decreasing, the equation of equal altitudes is negative. What difference will there be if the latitude and declination be of the same name, but the declination is equal to or exceeds the latitude?

Ex. 336. State the conditions under which the equation of equal altitudes vanishes. *A.* 1862.

Ex. 337. In the equation of equal altitudes find $M - N$, having given—

$$M = \tfrac{1}{30}d' \,.\, \tan. L \,.\, \text{cosec.} \tfrac{1}{2}\epsilon,$$

$$\text{and } N = \tfrac{1}{30}d' \,.\, \cot. \tfrac{1}{2}\epsilon \,.\, \cot. p,$$

where $p = 80°$ $\epsilon = 3\text{hrs.}$ $L = 60°$ $d = 12'$.

Royal Naval College, 1867.

Ex. 338. Describe accurately the determination at sea of the time of noon: and show how it serves to determine the longitude of the place. *Second B.A. and B.Sc. London*, 1877.

Ex. 339. After taking the observation of the altitude of the sun in the morning, calculate at what time *by the chronometer* the corresponding equal altitude must be observed. *Example:* A chronometer is about 17 minutes fast of mean time at the place; the time of the a.m. observation 10h. 46m. 57s. by chronometer; equation of time 4m. 10s. subtractive from apparent time. What time will the chronometer show at the equal altitude in the afternoon?

For Beaufort Testimonial, 1865.

Ex. 340. February 10th, 1887, in latitude 22° 18′ N., longitude 93° E., the sun had equal altitudes when the chronometer showed 3h. 34m. 4·36s. a.m. and 8h. 37m. 40·42s. a.m. Required the error of the chronometer on G. M. T. *H.* 1881.

Ex. 341. In determining the direction of the meridian at any permanent place of observation by means of two equal altitudes of the sun taken before and after noon, investigate by any method the correction to be applied for the sun's change in declination during the interval between the observations.

Second B.Sc. London, 1878.

CHAPTER XIII.

Compass errors—Definitions—Methods of finding compass errors—By meridian passages—By an amplitude—Objects to be used—Proof of formula — Rule — Example — Exercise — Length of day—The formula cos. $h = -$ cot. p . tan. l discussed—The time an object takes to rise—Proof of formula—Difference in length of morning and afternoon—Proof of formula—Example—Exercise—On twilight —Proof of formula—Example—Exercise—Examination.

THE TERRESTRIAL MERIDIAN is half the vertical circle passing through the zenith and the north and south points of the horizon.

THE MAGNETIC MERIDIAN is half the great circle in the heavens traced out by a vertical plane through the poles of a freely suspended magnet or compass needle which is not affected by any other force than the earth's.

VARIATION is the angle at the zenith between the terrestrial and magnetic meridians. It is usually measured by the arc of the horizon intercepted between its north or south point and the point where the magnetic meridian meets it.

DEVIATION is the angle at the zenith between the magnetic meridian and the great circle traced out by a vertical plane through the poles of a compass needle which is affected by surrounding objects, and is measured by the arc of the horizon intercepted between the magnetic and compass north.

LOCAL ATTRACTION is the error of the compass caused by the magnetic substances of the locality: such as the iron-bound coasts of Elba and Sweden; chains, mooring posts, and proximity of iron ships, &c., in harbours.

COMPASS ERROR is the whole error of the compass, and is composed of *variation*, *deviation*, and *local attraction.*

The *first* is caused by influences residing in the earth.

The *second* by those residing in the ship herself, and are therefore taken by the ship wherever she sails.

The *third* is caused by influences residing in the locality the

ship may be in, and are left behind when the vessel leaves that locality.

Variation is supposed to have been first noticed by Columbus in 1492. It changes with the geographical position of the observer, and also with a lapse of time. The latter changes are small (now about 7′ annually), and are constant for the time; and were it not for the iron, which is now so largely employed in the construction, the equipment, or the cargoes of our ships, compass error when once registered would serve for years to come by simply allowing the annual change. Iron is now so extensively used that there is scarcely a vessel that floats but her compasses are influenced by its action: and as the error caused by the iron is modified by the course the vessel is steering, by the heel she has, and by her geographical position, it is incumbent on every nautical man to be able to separate his errors and to find and apply the necessary corrections to his compasses. The principle to be employed is the same in every method used. The compass bearing of a heavenly body is taken, and the true bearing for the same instant is calculated: the difference between the two must necessarily be the error of the compass at the place of observation, for the position of the ship's head and the heel she had at the moment of observation. It is thus seen that *the error* thus found *includes both variation and deviation*, and as the first is given on all magnetic charts the latter can be deduced; but we must remind the student that deviation will vary at the same place for different courses and for different inclinations the vessel may have from the vertical. If the iron in the vessel exercise no influence on the compasses, the result obtained is variation only, and ought to agree with that registered on the chart for the ship's position.

METHODS FOR OBTAINING COMPASS ERRORS.—There are five methods in general use for determining compass errors.

(1) By the passages of bodies across the meridian and the prime vertical.

(2) By amplitudes.

(3) By altitude azimuths.

(4) By time azimuths.

(5) By the true bearing of a terrestrial object.

(1) BY MERIDIAN PASSAGES, &c.—When the time can be relied on, the simplest method is by meridian passages, for then all bodies not in the zenith must bear either due north or south. But local time at sea is rarely known with sufficient accuracy

to warrant the using of this method for *scientific* purposes; but a difference of five minutes of time in the meridian passage will give results within an eighth of a point, and time is generally known within this limit; and therefore, as few quartermasters can steer as near as this, the method of meridian passages may safely be used at sea.

Again, because an object, in a given time, has the least change in its bearing when near the prime vertical, and greatest when near the meridian, the time for observation of the crossing the prime vertical need not be so exact even as that for a meridian passage, and is easily calculated from

$$\text{Sin. hour angle} = \text{cot. lat.} \times \text{tan. dec.}$$

Then as the body is due east or west at that time, the error must be the difference the object bears by compass from the east or west point. This will be found a very convenient method in practice with the sun during the six summer months in high latitudes, and with stars which cross the prime vertical.

(2) BY AMPLITUDES.—*The amplitude* of a heavenly body is its angular distance from due east when rising and from due west when setting. By this method the compass error can be found by the sun twice every day in fine weather, and as often as necessary by stars in the night; but a limitation of latitude should be attended to, as will he hereafter shown.

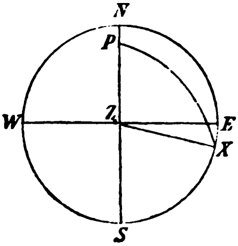
Because refraction causes bodies to appear higher than they actually are, and its effect is greatest when the object is rising or setting, being then about 33′, the bearing of an object should not be taken for an amplitude when its centre appears in the horizon, but when its centre is 33′ + dip above it. Taking the semidiameter of the sun at 16′, the altitude of his lower limb should be 17′ + dip when the amplitude of his centre is taken. The moon is a very unfavourable object to be used for amplitudes, because her horizontal parallax averages about two of her diameters, hence her centre is depressed two diameters whilst refraction raises it only about one; and, therefore, the moon's centre is nearly a diameter above when a spectator sees it in the horizon.

PROOF FOR AMPLITUDE.—The investigation of the rule for finding the true bearing of an object at its rising and setting is as follows:—

Let *N E S W* be the projection of the celestial concave on the plane of the horizon and the other letters denote the same

as in former diagrams. Join $Z X$ and $P X$, then, because the meridian cuts the horizon at right angles, therefore $Z N X$ is a right angle and $P N X$ is a right-angled triangle.

Fig. 39



Hence cos. $P X =$ cos. $P N$. cos. $N X$,
and cos. $N X =$ sec. $P N$. cos. $P X$;
but cos. $P X =$ cos. polar dist. $=$ sin. dec.
$P N$ is the latitude, and cos. $N X$ $=$ sin. $E X =$ sin. amplitude;
$\therefore$ sin. amp. $=$ sec. lat. $\times$ sin. dec.

In looking at the above formula, we see that sine amplitude increases as the secant of the latitude does; but, as the secant of an angle increases very rapidly as the angle becomes larger, and finally becomes infinite, so the sine of the amplitude must also increase, and by-and-by become greater than *one;* but, as the greatest sine an angle can have is unity; this shows that as the latitude increases the amplitude becomes impossible, that is, the object never rises nor sets.

At either of the poles an observer would see the fixed stars travel in circles parallel to the horizon, and the sun, moon, and planets when visible, nearly so. The nearer an observer is to the poles, the nearer the paths of celestial objects will coincide with circles parallel to the horizon: hence the bearings of all celestial bodies in high latitudes vary very much for small differences in altitudes. The error of the compass should, therefore, never be deduced from an amplitude when the latitude is high; but an azimuth, or the passing of an object over the prime vertical, should be used for that purpose.

Rule for an Amplitude.—(*a*) Find the Greenwich date corresponding to the time of observation.

(*b*) Correct the declination for the Greenwich date.

(c) Add together, log. secant of the latitude and log. sine of the declination, the sum is log. sine of the true amplitude. This must be marked E. at rising, and W. at setting of the object; and towards the N. or S. according to the declination.

d Under the true amplitude, set the bearing by compass, then if they be both N. or both S., their difference is the error; but if one be N. and the other S., their sum is the error of the compass.

(*e*) If the true amplitude be to the right of the magnetic

(viewed from the centre of the compass) the error is E., if to the left it is W.

(*f*) The difference between the compass error and the variation is the deviation.

Ex. 342. 1887, July 27th, at about 4h. 43m. a.m. apparent time at ship in latitude 42° 50′ N., longitude 138° 20′ E., the compass bearing of the sun was E.N.E. Find the error of the compass, and the deviation for the position of the ship's head, the variation for the geographical position being 3° 20′ W.

For Greenwich time.

App. time ship, July	26d.	16h.	43m.	0s.
Long. E	–	9	13	20
Greenh. mean time, July	26	7	29	40

Long. in time.

138° 20′ E.
4

60)553 20

In time 9h. 13m. 20s.

For sun's declination.

July 26th	19° 28′ 0·4″ N.
Correction	– 4 8
True dec.	19 23 52·4 N.

Variation of dec.

For 1h.	–	33·07
No. hrs.		7·5
Correction		248·025

For true amplitude and error.

Latitude		42° 50′ 0″		sec.	·134698
Declination		19 23 52		sin.	9·521301
True amplitude	E.	26 55 47	N.	sin.	9·655999
Compass bearing	E.	22 30 0	N.		
Compass error		4 25 47	W.		
Variation		3 20 0	W.		
Deviation		1 5 47	W.	*Answer.*	

STAR AMPLITUDE.—As the declinations of the fixed stars do not sensibly vary from day to day, neither a Greenwich date nor a correction for the declination is necessary.

EXERCISE XII.

Ex. 343. 1887, January 25th, at 4h. 23m. p.m. apparent time at ship in latitude 50° 35′ N., longitude 14′ 40″ W., the sun set by compass due West. If the variation from the chart

be 18° 20′ W., find the compass error and deviation for the position the ship's head was on at the time of taking the observation.

Ex. 344. 1887, December 21st, at about 6h. 11m. a.m. apparent time at ship in latitude 6° 10′ N., longitude 175° 27′ W., the sun bore S. 66° 44′ E. by compass at rising. Find the compass error, and if the variation from the chart be 20° 10′ E., find the deviation for the then position of the ship's head.

Ex. 345. 1887, May 31st. Find the deviation for the position of the ship's head if the sun set by compass N. ¼ E. at about 9h. 5m. mean time at ship in latitude 60° 30′ N., longitude 58° 15′ W. The variation from the chart was 52° 20′ W.

Ex. 346. 1887, September 28th, at about 5h. 53m. a.m. mean time at ship in latitude 23° 25′ S., longitude 12′ 40″ W., the sun's compass bearing at rising was E. ¾ N. Find the compass error and deviation for the position the ship's head was then on if the variation be 2¼ points W.

Ex. 347. 1887, November 24th. The sun set by compass due West at about 6h. 51m. p.m. mean time at ship in latitude 36° 43′ S., longitude 33° 29′ E. If the variation from the chart be 27° 15′ W., find the compass error and deviation for the time.

Ex. 348. 1887, October 28th, at about 6h. 40m. a.m. apparent time at ship, the sun rose by compass E. by S. ½ S. in latitude 36° 53′ N., longitude 152° 50′ E. Find the compass error and deviation if the variation be 2° 50′ E.

Ex. 349. 1887, December 22nd, at about 9h. 32m. a.m. apparent time at ship, the sun's compass bearing at rising was S. by E. ½ E. If the variation from the chart be 26° 20′ W., find the compass error and deviation for the then position of the ship's head in latitude 62° 20′ N., longitude 4° 8′ 30″ E.

Ex. 350. 1887, April 2nd, at about 6h. 9m. p.m. mean time at ship, the sun's bearing when setting was W. ¼ S. in latitude 14° 20′ N., longitude 165° 0′ 20″ E. If the variation from the chart be 9° 50′ E., find the compass error and deviation for the then position of the ship's head.

Ex. 351. 1887, January 21st, in latitude 53° 28′ N., longitude 3° 20′ E., Spica rose by compass S.E. ¼ S. If the deviation at that time was 7° 28′ W., what was the variation of the compass?

Ex. 352. If Arcturus set by compass W. ¼ S. on 23rd August, 1887 in latitude 5° 29′ N., longitude 158° 20′ W., find the

correction for the course and the deviation for the then position of the ship's head, the variation from the chart being 8° 45′ E.

Ex. 353. At a place on the Antarctic circle, what will be the bearing of the sun at rising at the winter solstice?

Royal Naval College, 1867.

Ex. 354. Define amplitude, and show how to find the variation of the compass by an observed amplitude of the sun. Prove the formula you employ. *E.* 1869.

Ex. 355. Prove the rule for finding the true bearing of the sun at rising or setting; and state what observations you must combine with this to find the error of your compass. What else would be necessary to deduce the variation of your compass, and whence is this obtained? *A.* 1871.

Ex. 356. What will be the compass bearing of the sun at setting at a place in latitude 46° 20′ N., when the declination is 17° 46′ S., variation 18° 30′ W., deviation 9° 10′ E.?

Final Ex. H.M.S. Britannia, 1875.

Ex. 357. In a given latitude l it is required to find the declination of the sun when he set S.S.W.

Royal Naval College, 1865.

Ex. 358. Show that the declination d of a star that rises N.E. may be calculated for a place in lat. l from the formula

$$\sin. d = \tfrac{1}{2}\sqrt{2} \cdot \cos. l.$$

For Beaufort Testimonial, 1865, *and H.* 1870.

Ex. 359. Why should the moon not be used for determining compass errors by an amplitude?

Ex. 360. In what parts of the world is the result obtained by an amplitude likely to be faulty? Give full reasons for your answer.

LENGTH OF THE DAY.

With the amplitude are connected the most interesting subjects of the rising and setting of celestial objects, the length of the day, duration of twilight, and the phenomena generally which are attendant on the earth's daily rotation. The student should be careful to notice in what follows how mathematical formulæ are applied to interpret physical facts. In the fundamental formula used in Nautical Astronomy,—

$$\begin{aligned}\text{Cos. } z &= \cos. p \cdot \cos. l' + \sin. p \cdot \sin. l' \cdot \cos. h \\ &= \cos. p \cdot \sin. l + \sin. p \cdot \cos. l \cdot \cos. h,\end{aligned}$$

we have when the object is in the horizon, that is at its true rising and setting, the zenith distance $z = 90°$; then

$$\cos. h = -\frac{\cos. p \,.\, \sin. l}{\sin. p \,.\, \cos. l}$$
$$= -\cot. p \,.\, \tan. l.$$

(*a*) *If p be less than* 90°, cot. p is positive, and therefore cos. h is negative, that is, h is greater than 90° or 6 hours. But when p is less than 90°, the latitude and declination are of the same name; hence at such times the object rises and sets more than 6 hours from the time of its crossing the meridian, and in the case of the sun the days will consequently be more than 12 hours in length.

(*b*) *If p equals* 90°, then cot. $p = 0$, and therefore cos. $h = 0$, that is, the hour angle is 90° or 6 hours. Hence, whatever the latitude of the observer, if the declination of the object be 0, it will rise and set 6 hours from the time of transit. In the case of the sun, therefore, the days will be exactly 12 hours long if refraction be neglected. This happens at the equinoxes.

(*c*) *If p be greater than* 90°, cot. p will be negative, cos. h will therefore become positive, and hence the hour angle less than 90° or 6 hours. But p is greater than 90° when the latitude and declination are of different names; therefore at such times the object rises and sets less than 6 hours from its transit. In the case of the sun, therefore, the days will be less than 12 hours in length.

(*d*) *If l equals* 0, tan. $l = 0$, and consequently cos. $h = 0$, or the hour angle is 90° or 6 hours. Therefore, if the observer be on the equator, all celestial bodies are 12 hours above and 12 hours below the horizon.

(*e*) *If l be so great and p so small that the right-hand side of the equation becomes numerically greater than unity*, then cos. h becomes impossible because the cosine of an angle can never exceed unity, that is, the object never rises and never sets. Thus we see the limits of cos. h are ± 1: with the $+$ sign, $h = 0$, or the body never rises, but in its diurnal rotation just grazes the horizon without going above it, and in the case of the sun, there is no daylight: with the $-$ sign, $h = 180°$ or 12 hours, that is, the object in its daily motion just touches the horizon without going below it, and the day is continuous. When cos. $h = -1$, then tan. $l =$ tan. p, or $l = p$. Hence, when the latitude equals or exceeds the polar distance, the object never sets. This is the case with the circumpolar stars (e.g. Ursæ Major in

our own latitude) with the sun at the solstices, and with the moon and planets at their greatest declination towards the elevated pole in high latitudes.

These remarks apply to the *true* rising and setting of a celestial object, or to the instant when its centre is on the horizon; but if we consider the time of rising the moment when its upper limb is on the horizon, then, owing to the apparent change in its position caused by dip, semidiameter, refraction, and parallax, its appearance to an observer is greatly modified. From these causes the true rising of the sun occurs when his lower limb is rather more than half a diameter above the visible horizon: with the moon, as already explained, the true rising takes place when the upper limb is about half a diameter below the visible horizon: and with the stars the atmosphere absorbs so much of their light that they are rarely visible at a less altitude than 5°. From the above it will be seen that the sun's rising is accelerated by refraction and dip, but retarded by parallax, and his setting is delayed by the two former, but accelerated by the latter. With the length of the day are connected many interesting problems, such as the time it takes an object with a visible disc to rise. This is shown as follows; and because the diameters of all the celestial objects are so small, the effects of refraction on the upper and lower limbs are considered equal. Beginning with the fundamental formula,

$$\text{Cos. } z = \text{cos. } p \text{ . cos. } l' + \text{sin. } p \text{ . sin. } l' \text{ cos. } h,$$

where the zenith distance and the hour angle are the only variables, we have

$$- \text{sin. } z \text{ . } dz = - \text{sin. } p \text{ . cos. } l \text{ . sin. } h \text{ . } dh$$

$$\therefore \qquad dh = \frac{\text{sin. } z}{\text{sin. } p \text{ . cos. } l \text{ . sin. } h} dz;$$

but z is so nearly equal to 90°, that we may consider sin. $z = 1$, and then cos. $h = -$ cot. p . tan. $l = -$ tan. dec. tan. lat.;

$$\text{hence } dh = \frac{dz}{\text{cos. dec. cos. lat. } \sqrt{1 - \text{tan.}^2 \text{dec. tan.}^2 \text{lat.}}} \text{ nearly}$$

$$= \frac{dz}{\sqrt{\text{cos. (lat. + dec.) . cos. (lat. } - \text{dec.)}}} \quad . \quad \text{I.}$$

Now, dh is the small increment in the hour angle corresponding to the small increment dz in the zenith distance; then, if D be the diameter of the object,

$$dz = D \text{ in seconds,}$$

and dh = number of seconds in arc the object will take to rise through a space equal to its diameter when on the horizon;

$\therefore$ 15 times number of seconds in time $= dh$;

$$\therefore \left.\begin{array}{l}\text{No. secs. in time}\\ \text{for objects to rise}\end{array}\right\} = \frac{\dfrac{D''}{15}}{\sqrt{\cos.\ (\text{lat.} + \text{dec.})\ .\ \cos.\ (\text{lat.} - \text{dec.})}}.$$

This formula may also be obtained without the use of the differential calculus. Thus:—

Let z_1 and h_1 be the zenith distance and hour angle when the upper limb is on the horizon,

and z_2 and h_2 be the same quantities when the lower limb is on the horizon;

then $\cos.\ z_1 = \cos.\ p\ .\ \cos.\ l' + \sin.\ p\ .\ \sin.\ l'\ .\ \cos.\ h_1$

$\cos.\ z_2 = \cos.\ p\ .\ \cos.\ l' + \sin.\ p\ .\ \sin.\ l'\ .\ \cos.\ h_2$;

$$\therefore \cos.\ z_1 - \cos.\ z_2 = \sin.\ p\ .\ \sin.\ l'\ (\cos.\ h_1 - \cos.\ h_2)$$

$$2 \sin. \frac{z_1 - z_2}{2}\ .\ \sin.\ \frac{z_1 + z_2}{2} = \sin.\ p . \sin.\ l' . 2 \sin.\ \frac{h_1 - h_2}{2} . \sin. \frac{h_1 + h_2}{2}.$$

Here $\frac{z_1 + z_2}{2} = 90°$, and $\frac{h_1 + h_2}{2}$ is so very near the hour angle when the centre is on the horizon that we may assume it $= h$ that hour angle; then—

$$\sin.\ \frac{z_1 - z_2}{2} = \cos.\ d\ .\ \cos.\ l\ \sin.\ h\ .\ \sin.\ \frac{h_1 - h_2}{2}$$

$$= \cos.\ d\ .\ \cos.\ l\ .\ \sqrt{1 - \cos.^2 h}\ .\ \sin.\ \frac{h_1 - h_2}{2}$$

$$= \sqrt{\cos.\ (l + d)\ .\ \cos.\ (l - d)}\ .\ \sin.\ \frac{h_1 - h_2}{2}$$

$$\therefore \frac{h_1 - h_2}{2} \sin.\ 1'' = \frac{\dfrac{z_1 - z_2}{2}\ .\ \sin.\ 1''}{\sqrt{\cos.\ (l + d)\ \cos.\ (l - d)}},$$

$$\text{or } \left.\begin{array}{l}\text{the difference in}\\ \text{the hour angle}\end{array}\right\} = \frac{\text{difference in zen. distance}}{\sqrt{\cos.\ (l + d)\ \cos.\ (l - d)}},$$

which is the same as the equation marked I. above. If the interval between the apparent and true times of rising be required, then for D we must write $33' +$ dip $-$ parallax reduced to seconds, and the formula becomes for the sun—

$$\left.\begin{array}{l}\text{No. seconds between}\\ \text{appt. and true sunrise}\end{array}\right\} = \frac{\dfrac{33' + \text{dip} - 9''}{15}}{\sqrt{\cos.\ (\text{lat.} + \text{dec.})\ .\ \cos.\ (\text{lat.} - \text{dec.})}}.$$

If the change in the declination of an object be considered, it is evident the eastern and western hour angles cannot be of the same magnitude; or, in the case of the sun, the mornings and afternoons cannot be of the same length. The investigation of the difference in the lengths of the morning and afternoon is as follows:—

Let θ be the difference in the declination of the sun at rising and setting, then at such times

$$\cos. h = -\cot. p \,.\, \tan. l,$$
$$\text{or} - \cos. h = \tan. \text{dec.} \,.\, \tan. l,$$

where the hour angle and the declination are the variables;

$$\therefore \sin. h \,.\, dh = \tan. l \,.\, \sec.^2 \text{dec.} \,.\, d\,(\text{dec})$$
$$dh = \frac{\sec.^2 \text{dec.} \tan. l}{\sin. h} \,.\, \theta$$
$$= \frac{\sec.^2 \text{dec.} \tan. l}{\sqrt{1 - \tan.^2 \text{dec} \tan.^2 \text{lat.}}} \,.\, \theta$$
$$= \frac{\sec. \text{dec.} \,.\, \sin. \text{lat.}}{\sqrt{\cos. (\text{lat.} + \text{dec.}) \,.\, \cos. (\text{lat.} - \text{dec.})}} \,.\, \theta.$$

Now, when θ is positive, that is when the declination is increasing, and therefore the polar distance decreasing, as it is from the winter to the summer solstice, the afternoons will be longer than the forenoons by $\dfrac{\sec. \text{dec.} \,.\, \sin. \text{lat.}}{\sqrt{\cos. (\text{lat.} + \text{dec.}) \,.\, \cos. (\text{lat.} \text{dec.})}} \,.\, \dfrac{\theta}{15}$ seconds of time, and shorter by that quantity for the remainder of the year. The equation of time, too, must be considered when comparing the lengths of the morning and afternoon.

Ex. 361. How long will it take the sun to rise out of the horizon at the Navigation School, Plymouth, in latitude 50° 22′ 25″ N., on Christmas Day, 1887, when his semidiameter is 16′ 18·1″ and declination 23° 24′ 40″ S.?

$$D = 16' \, 18{\cdot}1'' \times 2 = 32' \, 36{\cdot}2''$$
$$= 1956{\cdot}2''$$

$$\text{No. seconds in rising} = \frac{\dfrac{D}{15}}{\sqrt{\cos. (\text{lat.} + \text{dec.}) \,.\, \cos. (\text{lat.} - \text{dec.})}}$$
$$= \frac{130{\cdot}413}{\sqrt{\cos. 73° \, 47' \, 5'' \times \cos. 26° \, 57' \, 45''}}$$
$$= 261{\cdot}4$$
$$= 4\text{m. } 21{\cdot}4\text{s.} \quad \textit{Answer.}$$

The student who feels interested in this work can prove the above result by finding the hour angles when the zenith distance is $90° + 16' \ 18{\cdot}1''$, and also when it is $90° - 16' \ 18{\cdot}1''$. The difference in the two hour angles is the time the sun takes to rise.

EXERCISE XIII.

Ex. 362. At what time does the sun rise in latitude 51° N. when his declination is 22° 4′ 30″ N.? *For Lieutenant*, 1874.

Ex. 363. In latitude 50° 48′ N. find the difference in the longest and shortest days. *Royal Naval College*, 1865.

Ex. 364. To a spectator in the northern hemisphere the sun, whose declination = 15°, rises just two hours before noon: prove that the tangent of the latitude $= \frac{1}{2}\sqrt{3}\sqrt{\dfrac{1 + \frac{1}{2}\sqrt{3}}{1 - \frac{1}{2}\sqrt{3}}}$.
For Beaufort Testimonial, 1866.

Ex. 365. Obtain an expression for determining the time of the rising or setting of a heavenly body. In what latitude will the length of the day be twice as long as that of the night, when the sun's declination is 20° 20′ N.? *A.* 1872.

Ex. 366. In what latitude is the difference between the longest and shortest days exactly 6 hours? The sun's maximum declination may be assumed to be 23° 28′. *H.* 1881.

Ex. 367. In what latitude will the shortest day be just $\frac{3}{5}$ the longest? *Royal Naval College*, 1863.

Ex. 368. Explain why the days are longer in summer than in winter. Why is the time from sunrise to twelve o'clock not generally equal to the time from twelve o'clock to sunset? At what seasons of the year are these two periods most nearly equal? *Second B.A. and B.Sc. London*, 1862.

Ex. 369. At a certain place, when the sun's declination was $a°$ N., it rose one hour later than when it was $b°$ N., find an expression (adapted to logarithms) for computing the latitude.
For Beaufort Testimonial, 1864.

Ex. 370. Explain how the change in the length of the day at different seasons of the year is accounted for. What changes in the length of the day are to be observed at a place within the Arctic circle? *Second B.A. and B.Sc. London*, 1867.

Ex. 371. At a certain place, when the sun's declination was 10° N., it rose 52 minutes later than when it was 20° N. Required the latitude. *Royal Naval College*, 1863.

Ex. 372. Explain the causes of change of seasons, and of the different lengths of day and night. Show that the full moon in winter is longer above the horizon than in summer.

For Beaufort Testimonial, 1865.

Ex. 373. Given the refraction in the horizon $= r$: find an expression for calculating how much the rising of the sun is accelerated by it; the latitude of the place and sun's declination being given. *For Beaufort Testimonial*, 1864.

Ex. 374. Draw a spherical triangle that shall serve to determine the duration of the longest day in the year at any given place. State and explain the phenomena of day and night at the equator and poles respectively. In what portions of the earth's surface is the sun sometimes above and sometimes below the horizon for twenty-four hours and upwards?

Second B.A. and B.Sc. London, 1871.

Ex. 375. In what latitude north will the shortest day be just one-fourth the longest, declination $= 23° 28'$.

Royal Naval College, 1867.

Ex. 376. Find how long the sun will be in rising out of the horizon, his S. D. being $15' 57{\cdot}6''$, his declination $20° 10' 30''$ N., and his latitude $40° 20'$ S. *Royal Naval College*, 1872.

Ex. 377. On elementary principles of spherical astronomy, explain why all the year round the length of day at any latitude is equal to the length of night at the opposite latitude, and conversely. *Second B.A. and B.Sc. London*, 1876.

Ex. 378. Neglecting refraction, how would you calculate the time of rising of a star of which the R. A. and N. P. D. are given, the sidereal time of mean noon of the days before and after being also given. *Second B.A. and B.Sc. London*, 1879.

Ex. 379. Given that, on an unknown day, at an unknown place, the sun is just twelve hours above or twelve hours below the horizon of the place; state exactly what may be inferred from the single circumstance respecting the day, or the place, or both. *Second B.Sc. London*, 1877.

ON TWILIGHT.

Every one, at the setting of the sun, will ha
it does not become instantaneously dark when
below the horizon, but that it gets d
night draws on. This is caused by t
its reflection from particles of water

suspended in the atmosphere. It has been found from observation that some of this diffused light reaches the observer until the sun sinks 18° below the horizon, when true night begins. *The interval between sunset and night is called twilight.* What we have said of evening twilight applies equally to morning twilight in the reverse order. As soon as the sun in rising comes within 18° of the horizon, morning twilight begins, and this continues until sunrise. From this it is evident there is no true night at those places where he does not sink 18° below the horizon; that is, at those places which (at the summer solstice) are above $48\frac{1}{2}°$ of latitude. As the elevation of the nearest pole measures the latitude of the place, therefore at all places where the latitude + 18° is greater than the polar distance there will be no true night. Thus at the Navigation School, Plymouth, in latitude 50° $22\frac{1}{2}'$ N., there is no night so long as the north polar distance is less than 50° $22\frac{1}{2}'$ + 18° = 68° $22\frac{1}{2}'$; that is so long as the sun's declination is as much as 21° $37\frac{1}{2}'$ N., or from 29th May to 14th July.

The investigation of the time of the beginning or the end of twilight is easily made by writing $(90 + \delta)$ for z in the fundamental formula; where δ is the depression below the horizon or 18°. Then

$$\cos. h = \frac{\cos. (90 + \delta) - \cos. p \ \cos. l'}{\sin. p \,.\, \sin. l'}$$

$$1 - \cos. h = 1 + \frac{\cos. p \,.\, \sin. l + \sin. \delta}{\sin. p \,.\, \cos. l}$$

$$2 \sin.^2 \frac{h}{2} = \frac{\sin. (p + l) + \sin. \delta}{\sin. p \,.\, \cos. l}$$

$$= \frac{2 \sin. \frac{p + l + \delta}{2} \,.\, \cos. \frac{p + l - \delta}{2}}{\sin. p \,.\, \cos. l}.$$

[illegible]

[illegible] $. \sin. s \,.\, \cos. (s - \delta)$.

[illegible] ebraical sum of the correc-
[illegible] x for δ, may also be used to
[illegible] ting of the sun and stars, as
[illegible] f twilight. Now from what

has gone before the time of the true rising is obtained from

$$\cos. h. = - \cot. p . \tan. l.$$

The difference of the times, therefore, obtained by these two formulæ when applied to the sun is the duration of twilight, which is thus found to depend entirely on the sun's declination and the latitude of the observer, because δ is constant. Hence to find the duration of twilight is simply to find the interval of time the sun takes to alter his zenith distance from 108° to 90° in the morning, and from 90° to 108° in the evening. It is easily seen the shortest duration of twilight is at the equator when the sun's declination is 0, for then he rises perpendicularly to the horizon and hence takes the time necessary for him to move through 18°, that is 1h. 12m.

In the case of a fixed star, its declination is constant, hence either of the above formulæ can be at once applied; but in the case of the sun, moon, and planets whose declinations are variable, a date as near as possible to the correct time of the phenomenon must be judged, and after an approximate time has been calculated, the whole must be repeated with the corresponding Greenwich date for the correction of the declination.

Ex. 380. How long will twilight last at the Navigation School, Plymouth, in latitude 50° 22′ 25″ N., longitude 4° 7′ 16·5″ W., on 2nd May, 1887, at sunrise?

Approx. time sunrise 4h. 43m. a.m. Approx. dec. 15° 16′ N.

$$
\begin{aligned}
-\cos. h &= \cot. p \,.\, \tan. l. \\
&= \cot. 74° 44' . \tan. 50° 22\tfrac{1}{2}' \\
\therefore \quad h &= 109° 15' \\
&= 7h.\ 17m.
\end{aligned}
$$

For Greenwich date.

Approx. time at ship, May 1d.	16h.	43m.	0s.
Long. in time, W.	+	16	29
Greenwich app. time, May 1	16	59	29

Long. in time.

Long. 4° 7′ 16·5″
= 16m. 29s.

For sun's declination.

May 2	=	15° 21′ 52″ N.
Correction		− 5 13
True dec.		15 16 39 N.
N. P. D.		74 43 21

Correction for declination.

In 1 hr. − 44·77
7

60)313·39

In 7 hours − 5·13

d	18° 0′ 0″		
l	50 22 25	sec.	·195330
p	74 43 21	cosec.	·015625
	2)143 5 36		
s	71 32 48	sin.	9·977078
$s-d$	53 32 48	cos.	9·773895
			2)19·961928
$\frac{h}{2}$	73° 9′ 34·4″	sin.	9·980964

$$-\cos. h = \cot. p \,.\, \tan. l$$
$$= \cot. 74° 43' 11'' \,.\, \tan. 50° 22' 25''$$
$$h = 109° 15' 52{\cdot}5''$$
$$= 7 \text{ hrs. } 17\text{m. } 3{\cdot}5\text{s.}$$

∴ hour angle 146° 19′ 8·8″
= 9h. 45m. 16·6s.

Hour angle beginning of twilight	= 9h.	45m.	16·6s.
,, ending ,,	= 7	17	3·5
∴ Duration of twilight	= 2	28	13·1

EXERCISE XIV.

Ex. 381. Assuming an equation in oblique angled spherical trigonometry, investigate a formula connecting amplitude, declination, and hour angle when the heavenly body is in the horizon. *Royal Naval College*, 1872.

Ex. 382. Although the 21st of June is the longest day of the year, the sun sets later in the evening on the 22nd, and later still on the 23rd. Explain this.
Second B.A. and B.Sc. London, 1879.

Ex. 383. Investigate formulæ for finding the times of the rising and setting of a heavenly body; also for the duration of twilight.

In what latitude south is the time of sunset half an hour later when the sun's declination is 5° S. than on the day when his declination is 5° N? *H.* 1876.

Ex. 384. Show how to find the length of the day at a given place. What is the length of the day at a place in latitude 43° S. on 14th May, 1887? What will be the duration of twilight at the same place on the same day? *A.* 1869.

Ex. 385. What is the cause of twilight? Find the duration of twilight at a place in latitude 30° N. when the sun's declination is 20° N. *Royal Naval College*, 1868.

Ex. 386. Compare the length of the longest and shortest days in a given latitude; investigate the conditions of the sun not setting during 24h., and also of twilight filling the interval between his setting and rising. *H.* 1873.

Ex. 387. Find the time elapsed between the beginning of twilight in the morning and the end of twilight in the evening at a place in latitude 35° N. when the sun's declination is 10° S. *Royal Naval College*, 1869.

Ex. 388. Find the time of daybreak in latitude 50° N., the declination of the sun being 20° S. (At daybreak the sun is 18° below the horizon.) *Royal Naval College*, 1866.

Ex. 389. Find the duration of twilight in latitude 63° N. on the shortest day. *H.* 1882.

Ex. 390. Find the sidereal time when Spica will bear due west at a place in latitude 40° 30′ S. *H.* 1883.

Ex. 391. Explain to what causes the phenomena of refraction and twilight are respectively due. Find how much the time of a star's rising is altered by refraction. Investigate the condition of morning twilight commencing when evening twilight ends. *H.* 1877.

CHAPTER XIV.

The azimuth of a body—Best time for observing the azimuth—The altitude azimuth—Proof of formula—Modification used in the Royal Navy—Rule—Example—Exercise—The time azimuth—Proof of formula—Rule—Example—A second method for time azimuth—Proof of formula—Rule—Example—To find the true bearing of a terrestrial object—Proof of formula—Example—Exercise—Examination.

The Azimuth of a body is the angle at the zenith between a vertical circle through the object and the meridian of the observer. Or

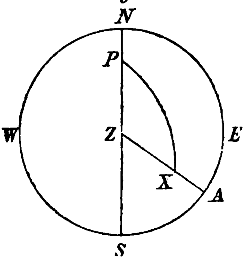
The Azimuth is the arc of the horizon intercepted between that point in it cut by a vertical circle through the object and its north or south point. It has the same meaning as is expressed by the bearing of the object. Theoretically, observations for azimuth should be made when the object is changing its bearing the quickest, that is when the object is moving parallel to the horizon, and therefore is nearest to the meridian or at its maximum altitude. But owing to the motion of the vessel, and also to the fact that the wire in the sight vane of the azimuth compass is seldom perfectly vertical, the instrumental errors introduced when the object has a great altitude are probably greater than those which can arise from the motion of the object in the heavens. It will, therefore, in practice be found more advantageous to take observations for compass errors when the body is comparatively low (not exceeding 15° or 20°), and therefore not in the position warranted by theory.

The Altitude Azimuth.—The computation of the true azimuth of a heavenly body depends either on its altitude or its hour angle. When the former is used it is called an *altitude azimuth*, when the latter, *a time azimuth*. For altitude azimuths,

the time at ship (which is the most difficult element to be determined) need not be so exact as when time azimuths are used: hence altitude azimuths are generally preferred in practice. The method *pursued* is as follows: one observer takes the altitude, then the bearing, and lastly a second altitude of the object in quick succession, while his assistant notes the times of the three observations. The change in altitude is at once deduced for the interval between the times of taking the altitudes, and from this the altitude at the instant of observing the bearing is found by simple proportion. If one of the observers be accustomed to counting seconds, the two observers should take the altitude of the body and its bearing by compass at the same moment, and the time should be noticed directly after. Then because the change in declination is so slow, no material error can arise in the true bearing calculated from the time of observation being a few seconds wrong.

Borda's investigation of the rule for altitude azimuths is as follows:—

Fig 40.



Let $N\ E\ S\ W$ be the projection on the plane of the horizon, $N\ S$ the meridian of the observer, P the elevated pole, Z the zenith of the observer, X an object whose altitude is $A\ X = a$, then $P\ X$ is the polar distance $= p$, $Z\ X$ its zenith distance $= z$, $P\ Z$ the colatitude $= l'$, and the latitude $= l$.

In the figure we have supposed the north pole the elevated one, and therefore the observer in north latitude: and for reasons seen in the following proof the azimuth A is reckoned from the south point of the horizon when the observer is in north latitude, and from the north point in south latitude.

In the spherical triangle $Z\ P\ X$, for an altitude azimuth, we have given the three sides $p\ .\ l'$ and z, to find the angle $P\ Z\ X$, and thence its supplement $S\ Z\ A$.

$$\text{Now } \cos. p = \cos. l' . \cos. z + \sin. l' . \sin. z . \cos. P\,Z\,X;$$

$$\therefore \cos. P\,Z\,X = \frac{\cos. p - \cos. l' . \cos. z}{\sin. l' . \sin. z} \quad . \quad . \quad . \quad . \quad \text{I.}$$

$$\text{Hence} \quad -\cos. A = \frac{\cos. p - \sin. l . \sin. a}{\cos. l . \cos. a}$$

$$1 - \cos. A = 1 + \frac{\cos. p - \sin. l . \sin. a}{\cos. l . \cos. a}$$

$$= \frac{\cos. (l + a) + \cos. p}{\cos. l . \cos. a}$$

$$\therefore \ 2 \sin.^2 \frac{A}{2} = \frac{2 \cos. \dfrac{l + a + p}{2} . \cos. \dfrac{l + a - p}{2}}{\cos. l . \cos. a}$$

$$\text{Let } \frac{l + a + p}{2} = S; \text{ then } \frac{l + a - p}{2} = S - p,$$

$$\text{and } \sin. \frac{A}{2} = \sqrt{\sec. l . \sec. a . \cos. S . \cos. (S - p)}. \quad \text{II.}$$

From formula II. the following rule is taken :—

RULE (*a*) Find the Greenwich time of observation.

(*b*) Correct the declination and get the polar distance.

(*c*) Correct the observed altitude and get the true altitude.

(*d*) Then add together the polar distance, the latitude and altitude, take half the sum (S) and find the difference between the polar distance and the half-sum ($S - p$).

(*e*) Add together log. sec. latitude, log. sec. altitude, rejecting the 10s. from each index, log. cos. of the half-sum, and log. cos. of the difference between the half-sum and the polar distance. Divide the sum of the logs. by two, this gives log. sine of half the true azimuth.

$$\sin. \frac{A}{2} = \sqrt{\sec. l . \sec. a . \cos. S . \cos. (S - a)}.$$

(*f*) Double the half azimuth gives the true azimuth, reckoned from south in north latitude and from north in south latitude : towards the east when the object is increasing in altitude, and towards the west when it is decreasing in altitude.

(*g*) Place the bearing by compass under the true azimuth, then the difference between these is the compass error; and is east or west according as the true azimuth is to the right or left of the compass bearing.

(*h*) If the deviation be required for the direction the ship's

head had when the observation was made, the variation from the Admiralty chart for the ship's position should be placed under the compass error. The difference between these is the deviation, always reckoning from the variation towards the compass error.

Ex. 392. 1887, January 6th at 3h. 46m. 5s. p.m. mean time at ship in latitude 52° 45′ S., longitude 56° 56′ 15″ E., the sun's bearing by compass was N.W. by W. at the time the altitude of his U. L. was 39° 0′ 40″. The index correction for the sextant was + 3′ 36″, the height of the eye above the sea was 20 feet. If the variation was 35° 50′ W., find the compass error and the deviation for the position the ship's head was on at the time.

For Greenwich time.

Mean time at ship, Jan.	6d.	3h.	46m.	5s.
Long. in time	—	3	47	45
Greenwich mean time, Jan.	5	23	58	20

Long. in time.

Long. 56° 56′ 15″ E.
4
227 45 0
In time 3h. 47m. 45s.

For polar distance.

Declination, Jan. 6	22° 30′ 9·4″ S.
Correction	+ ·5
True declination	22 30 10 S.
S. P. D.	67 29 50

Correction for declination.

For 1h. + 18·06″
„ 1m. 40s. = ·028
14448
3612
·50568

To correct the altitude.

Observed alt. U. L.	39° 0′ 40″
Index error	+ 3 36
	39 4 16
Dip	— 4 24
	38 59 52
Semidiameter	— 16 18·2
	38 43 33·8
Refrac. and parallax	— 1 3
True altitude	38 42 30

For compass error, &c.

(p)	67° 29′ 50″		
(l)	52 45 0	sec.	·218034
(a)	38 42 30	sec.	·107717
	2)158 57 20		
S	79 28 40	cos.	9·261541
$S-p$	11 58 50	cos.	9·990435
			2)19·577727
$\frac{A}{2}$	37° 57′ 3″	sin.	9·788864
	2		

True azimuth	N. 75 54 6 W.
Compass bearing	N. 56 15 0 W.
Compass error	19 39 6 W.
Variation	35 50 0 W.
Deviation	16 10 54 E.

In the Royal Navy, formula I. is somewhat modified and is deduced from equation I. as follows:—

$$1 - \cos. PZX = \frac{\sin. l' . \sin. z + \cos. l' . \cos. z - \cos. p}{\sin. l' . \sin. z}$$

$$= \frac{\cos. l . \cos. a + \sin. l \sin. a - \cos. p}{\cos. l . \cos. a}$$

$$= \frac{\cos. (a \backsim l) - \cos. p}{\cos. l . \cos. a}$$

$$2 \sin.^2 \frac{PZX}{2} = \frac{2 . \sin. \left(\frac{p}{2} + \frac{a \backsim l}{2}\right) . \sin. \left(\frac{p}{2} - \frac{a \backsim l}{2}\right)}{\cos. l . \cos. a};$$

$$\therefore \sin. \frac{Z}{2} = \sqrt{\sec. a . \sec. l . \sin. \left(\frac{p}{2} + \frac{a \backsim l}{2}\right) \sin. \left(\frac{p}{2} - \frac{a \backsim l}{2}\right)} \quad \text{III.}$$

Adapting this to a table of haversines as in finding the hour angle—

$$\text{haver.}\, Z = \sqrt{\sec. a . \sec. l . \text{haver.} \{p + (a \backsim l)\} . \text{haver.} \{p - (a \backsim l)\}}.$$

The student should notice that in this case we have found the angle PZX and not its supplement, hence the azimuth must be reckoned of the same name as the latitude.

The use of the above formula III. may be seen in working the last example with it.

Taking the data as there found

$a =$	38°	42′	30″	sec.	·107717
$l =$	52	45	0	sec.	·218034
$a \backsim l =$	14	2	30		
$\frac{a \backsim l}{2} =$	7	1	15		
$\frac{p}{2} =$	33	44	55		
$\frac{p}{2} + \frac{a \backsim l}{2} =$	40	46	10	sin.	9·814924
$\frac{p}{2} - \frac{a \backsim l}{2} =$	26	43	40	sin.	9·652973
					19·793648
$\frac{Z}{2}$	52	2	58	sin.	9·896824

True azimuth S. 104 5 56 W., or N. 75° 54′ 4″ W.

EXERCISE XV.

Ex. 393. 1887, March 20th, at 7h. 16m. 37s. a.m. mean time at ship in latitude 41° 30′ N., longitude 126° 40′ W., the observed altitude of the sun's U. L. when bearing East by compass was 13° 0′ 30″; height of eye 27 feet; index error + 10′ 32″. If the variation at the place was 18° 10′ E., find the compass error and the deviation for the position the ship's head had at the time.

Ex. 394. 1887, November 15th, at 5h. 19m. 30s. p.m. mean time at ship in latitude 35° 20′ S., longitude 35° 10′ W., the sun bore by compass West. The observed altitude of the sun's L. L. at the time was 15° 28′ 50″, the sextant having no error; height of eye 20 feet. Required the deviation for the direction the ship's head had at the time, if the variation was 1° 20′ W.

Ex. 395. 1887, May 14th, at 6h. 18m. 36s. a.m. apparent time at ship, the sun's bearing by compass was E. ½ N., when the observed altitude of his U. L. was 18° 1′; height of eye 24 feet; index error − 9′ 51″. The latitude was 52° 20′ N., and longitude 129° 10′ W. If the variation was 25° 40′ E., find the compass error and the deviation for the position the ship's head had at the time.

Ex. 396. 1887, March 20th, mean time at ship 4h. 11m. 5s. p.m. in latitude 37° 22′ S., longitude 168° 20′ E., the sun bore by compass W. by N. ¾ N., when the observed altitude of his U. L. was 23° 21′ 30″; height of eye 14 feet; index error −1′ 30″. Required the compass error and the deviation for the position of the ship's head at the time of the observation, variation 13° 20′ E.

Ex. 397. 1887, April 14th, mean time at ship 13d. 14h. 34m. 58s. in latitude 35° 20′ S., longitude 40° 28′ E., the compass bearing of Spica was N. 70° W; the observed altitude of the star was 45° 37′ 20″; index correction − 34″; height of eye 23 feet; variation from the chart 26° 20′ W. Find the deviation for the position of the ship's head when the altitude was taken.

Ex. 398. 1887, May 11th, at 12h. 10m. 54·5s. mean time at ship in latitude 47° 16′ S., longitude 28° 2′ 30″ W., the observed bearing of the moon by compass was E. by N. ½ N.; the observed altitude of the moon's L. L. was 34° 26′ 30″; index error for the sextant + 1′ 37″; height of eye above the sea

26 feet; the variation taken from the chart was 4° 50′ W. Required the error of the compass and its deviation for the direction the ship's head had at the time of taking the altitude.

THE TIME AZIMUTH.

The horizon is often obscured by fog or haze, so that it is not always possible to take an altitude at sea; and threatening weather may render it prudent to obtain the error of the compass before "*a dirty time*" sets in. A time azimuth may then be used for the purpose; which consists in taking the bearing of the object and noticing either the time by chronometer or correct apparent time at ship at the instant of observation. Because refraction takes place in a vertical circle, it cannot affect the *bearing* of a heavenly body as taken by the compass; and as by this method the altitude is not used, it has the great advantage of being free from the errors introduced by atmospheric refraction.

Let the projection and figure be the same as for the altitude azimuth: then in the spherical triangle PZX are known the colatitude l', the polar distance p, and the hour angle $ZPX = h$, or two sides and the included angle to find the angle PZX.

This is given at once by Napier's analogies.

$$\left.\begin{array}{l}(1)\ \tan.\tfrac{1}{2}(Z+X) = \cos.\tfrac{1}{2}(p-l')\,.\,\sec.\tfrac{1}{2}(p+l')\,.\,\cot.\dfrac{h}{2} \\ (2)\ \tan.\tfrac{1}{2}(Z \backsim X) = \sin.\tfrac{1}{2}(p-l')\,.\,\mathrm{cosec.}\tfrac{1}{2}(p+l')\,.\,\cot.\dfrac{h}{2}\end{array}\right\},$$

whence Z and X can each be found. It should be borne in mind that *the greater angle is always opposite to the greater side*; that is $\frac{1}{2}(Z+X)$ and $\frac{1}{2}(p+l')$ *are of the same affection; and that Z is reckoned from the elevated pole, and will, therefore, be of the same name as the latitude.*

The rule is as follows:—

RULE (*a*) Find the Greenwich time of observation.

(*b*) Correct the declination and find the polar distance (p).

(*c*) Find the hour angle thus:—

(1) *If the object be the sun*, the apparent time at place is the hour angle if it be p.m.; but if it be a.m., the apparent time at place subtracted from twenty-four hours is the hour angle.

(2) *If it be any other object*, the right ascension of the mean sun and of the object must be found and corrected, then

W. hr. $\angle$ = R. A. mean sun + mean time at place — R. A. of object.

Take half the hour angle thus found $\left(\frac{h}{2}\right)$.

(*d*) Then (1) $\tan.\frac{1}{2}(Z+X)=\cos.\frac{1}{2}(p-l').\sec.\frac{1}{2}(p+l').\cot.\frac{h}{2}$.

The supplement of this angle must be used if $\frac{1}{2}(p+l')$ be greater than 90°.

(2) $\text{Tan}.\frac{1}{2}(Z \backsim X)=\sin.\frac{1}{2}(p-l').\text{cosec}.\frac{1}{2}(p+l').\cot.\frac{h}{2}$.

(*e*) Add $\frac{1}{2}(Z+X)$ and $\frac{1}{2}(Z \backsim X)$ together when the polar distance is greater than the colatitude, but subtract $\frac{1}{2}(Z \backsim X)$ from $\frac{1}{2}(Z+X)$ when the polar distance is less than the colatitude. This must be marked of the same name as the latitude, east when the object is increasing in altitude, and west when the object is decreasing in altitude.

(*f*) The problem is completed as before.

Ex. 1887, January 6th, at 3h. 46m. 5s. p.m. mean time at ship in latitude 52° 45′ S., longitude 56° 56′ 15″ E.; find the true bearing of the sun. If the compass bearing be N.W. by W., and the variation be 35° 50′ W., find also the deviation for the position of the ship's head at the time of taking the bearing.

For Greenwich time.

Mean time at ship, Jan.	6d.	3h.	46m.	5s.
Long. in time	–	3	47	45
Greenwich mean time, Jan.	5	23	58	20

Long. in time.

Long. 56°	56′	15″
		4
227	45	0

In time 3h. 47m. 45s.

For polar distance.

Declination, Jan. 6	22°	30′	9·4″	S.
Correction		+	·5	
True declination	22	30	10	S.
S. P. D.	67	29	50	

Correction for declination.

For 1h.	+ 18·06
„ 1m. 40s.	·028
	14448
	3612
	·50568

For equation of time.

E. T. Jan. 6	–	6m.	3·53s.
Correction		–	·03
True E. T.	–	6	3·5

Correction for E. T.

For 1hr.	– 1·095
„ 1m. 40s.	·028
	·030660

For half-hour angle.		*For other data.*	
Mean time at ship	3h. 46m. 5s.	Lat.	52° 45′ 0″
Equation of time	− 6 3·5	Colat.	37 15 0
App. time	3 40 1·5	P. D.	67 29 50
	60	$p+l'$	104 44 50
	4)220 1·5	$p-l'$	30 14 50
Hour angle	55 0 22	$\frac{p+l'}{2}$	52 22 25
Half-hour angle	27 30 11	$\frac{p-l'}{2}$	15 7 25

$$Tan.\frac{Z+X}{2} = cos.\frac{p-l'}{2}\ sec.\frac{p+l'}{2}\ cot.\frac{h}{2}$$

$$Tan.\frac{Z \backsim X}{2} = sin.\frac{p-l'}{2}\ cosec.\frac{p+l'}{2}\ cot.\frac{h}{2}$$

$\frac{p-l'}{2}$	15° 7′ 25″	cos. 9·984692	sin.	9·416478
$\frac{p+l'}{2}$	52 22 25	sec. ·214307	cosec.	·101270
$\frac{h}{2}$	27 30 11	cot. ·283466	cot.	·283466
$\frac{Z+X}{2}$	71 46 32	tan. 10·482465		
$\frac{Z-X}{2}$	32 19 21	tan.		9 801214
Z	S 104 5 53 W.			

True azimuth	N. 75° 54′ 7″ W.
Compass bearing	N. 56 15 0 W.
Compass error	19 39 7 W.
Variation	35 50 0 W.
Deviation	16 10 53 E.

The student will have noticed this is the same question as was solved by means of an altitude azimuth, and the same answer is obtained.

Another method for determining the true bearing of a heavenly body from the time, is by using a subsidiary angle. The investigation is as follows, and comes from a well-known formula in spherical trigonometry (see Todhunter, Art. 43, p. 20):—

$$\cot. PX . \sin. PZ = \cot. PZX . \sin. ZPX + \cos. PZ . \cos. ZPX;$$

or if A be the azimuth reckoned from the elevated pole, and h, d, and l be respectively the hour angle, declination, and latitude, then

$$\tan. d \,.\, \cos. l = \cot. A \,.\, \sin. h + \sin. l \,.\, \cos. h$$

$$\cot. A = \frac{\tan. d \,.\, \cos. l - \sin. l \,.\, \cos. h}{\sin. h}.$$

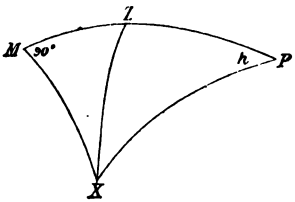
If now we assume $\tan. x = \cot. d \,.\, \cos. h$, we get

$$\cot. A = \frac{\cos. h \,.\, \cot. x \,.\, \cos. l - \sin. l \,.\, \cos. h}{\sin. h}$$

$$= \frac{\cos. h\,(\cos. x \,.\, \cos. l - \sin. x \,.\, \sin. l)}{\sin. h \,.\, \sin. x}$$

$$= \cot. h \,.\, \operatorname{cosec}. x \,.\, \cos. (x + l) \quad . \quad . \text{(A)}$$

Fig 41



This result may also be obtained as follows :—

From X draw a perpendicular $X M$ to meet the meridian $P M$ in M.

In the triangle $M P X$ let $M X = y$

$$\cos. h = \tan. x \,.\, \cot. p,$$

or $\tan. x = \cos. h \,.\, \text{cotan}. d$. I.

Again— $\sin. x = \cot. h \,.\, \tan. y$;

$\therefore \tan. y = \sin. x \,.\, \tan. h$ . . . . (*a*)

In the triangle $M Z X$—

$$\sin. M Z = \cot. (180 - A) \,.\, \tan. y.$$

From (*a*) $\sin. (x - l') = - \cot. A \,.\, \sin. x \,.\, \tan. h$.

Hence $\cot. A = \cot. h \,.\, \operatorname{cosec}. x \,.\, \cos. (x + l)$. (A),

which is the same result as before.

From the formulæ we deduce the following rule :—

RULE (*a*) From the data in the question, correct the declination and obtain the hour angle.

(*b*) Add together :—log. cos, hour angle, and log. cotan. declination ; the sum is log. tan., of an arc called x ;

i.e. $\tan. x = \cos. h \,.\, \cot. d$.

(*d*) Add together :—log. cot, hour angle, log. cosec. x and log. cos. the sum of x and the latitude; the result is log. cot, of the true azimuth of the same name as the latitude, if the sum of x and l be less than 90° ; but of a contrary name to the latitude if the sum of x and l be greater than 90° ;

i.e. $\cot. A = \cot. h \,.\, \operatorname{cosec} x \,.\, \cos. (x + l)$.

Ex. 399. 1887, January 6th, at 3h. 46m. 5s. p.m. mean time at ship in latitude 52° 45′ S., longitude 56° 56′ 15″ E., find the

deviation of the compass for the position of the ship's head, if the compass bearing be N.W. by W., and the variation 35° 50′ W.

Using the data already found—

$$\tan x = \cot. d \,.\, \cos. h$$

$$\cot. Z = \cot. h \,.\, \text{cosec.}\, x \,.\, \cos. (x + l)$$

h	55° 0′ 22″	cos. 9·758525	cot. 9·845128	
d	22 30 10	cot. ·382716		
x	54 9 24·6	tan. 10·141241	cosec. ·091182	
l	52 45 0			
$x + l$	106 54 24·6		cos. 9·463618	
True azimuth	N. 75° 54′ 7″ W.		cot. 9·399928	
Compass bearing	N. 56 15 0 W.			
Compass error	19 39 7 W.			
Variation	35 50 0 W.			
Deviation	16 10 53 E.			

The same answer as by the other methods.

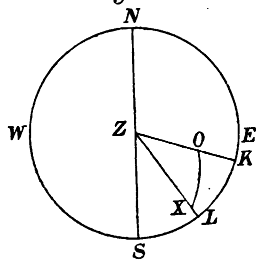
To facilitate work at sea, a very complete set of tables has been calculated by Mr. Burdwood, R.N., of the Hydrographic Department of the Admiralty. The arguments are the same as are used for computation, viz. the latitude, the declination, and the apparent time. The true azimuth is given for intervals of four minutes to every degree of declination and every degree of latitude as far as 60°. An interpolation must be used for intermediate times, latitudes, and declinations. The saving of time is more nominal than real when a near approximation to the true azimuth is required; but for sea purposes these tables will be found useful.

To find the True Bearing of a Terrestrial Object.

A knowledge of the correct bearing of a point of land or other object is frequently necessary when surveying a coast either for laying down the position of other objects, or for finding the deviation of the compass by swinging the ship. In surveying a strange coast the bearing of an object by the compass is of no practical utility, because there is no means of knowing at first what effect (if any) the iron in the composition of the rocks and soil has on the needle, and hence in laying down the direction of a base line it must always be done independent of the com-

pass. This is effected by first obtaining the true bearing of a heavenly body by either an altitude azimuth or a time azimuth; and as the altitude of the body is required for subsequent calculations, the former perhaps is preferable. The true bearing of the terrestrial object is then found as follows:—

Fig 42.



Let $N\ E\ S\ W$ be the horizon, Z the zenith, X the heavenly body, and O the object whose true bearing is required. The angle $S\ Z\ L$ is known by calculating the azimuth of X: then if the altitudes of $X = a$ and $O = a'$ be taken at the same time as the distance $X\ O = d$ is observed, we have $Z\ X = 90 - a$ and $Z\ O = 90 - a'$.

Let B be the difference in the bearings between X and O, so that $X\ Z\ O = B$.

$$\text{Then } \cos.\ X\ O = \cos.\ Z\ X\ .\ \cos.\ Z\ O + \sin.\ Z\ X\ .\ \sin.\ Z\ O\ .\ \cos.\ X\ Z\ O,$$

$$\text{or } \cos.\ d = \sin.\ a\ .\ \sin.\ a' + \cos.\ a\ .\ \cos\ a'\ .\ \cos.\ B \quad .\ .\quad \text{I.}$$

$$\cos.\ B = \frac{\cos.\ d - \sin.\ a\ .\ \sin.\ a'}{\cos.\ a\ .\ \cos.\ a'}$$

$$1 + \cos.\ B = \frac{\cos.\ a\ .\ \cos.\ a' - \sin.\ a\ .\ \sin.\ a' + \cos.\ d}{\cos.\ a\ .\ \cos.\ a'}$$

$$2\cos.^2 \frac{B}{2} = \frac{\cos.\ (a + a') + \cos.\ d}{\cos.\ a\ .\ \cos.\ a'}$$

$$= \frac{2\cos.\ \dfrac{a + a' + d}{2}\ .\ \cos.\ \dfrac{a + a' - d}{2}}{\cos.\ a\ .\ \cos.\ a'};$$

$$\therefore \cos.\ \frac{B}{2} = \sqrt{\sec.\ a\ .\ \sec.\ a'\ .\ \cos.\ S\ .\ \cos.\ (S - d)} \quad .\ \text{II.,}$$

$$\text{where } S = \frac{a + a' + d}{2}.$$

If the object be in the horizon, a' will be zero, and equation I. becomes—

$$\cos.\ d = \cos.\ a\ .\ \cos.\ B\ ;$$

$$\therefore \cos.\ B = \cos.\ d\ .\ \sec.\ a \quad .\ .\ .\ .\quad \text{III.}$$

This result may also be obtained directly, and is left as an exercise to the student.

In the above formulæ we must use the *apparent altitudes*

because we are finding the bearing of O on the surface of the earth, and the true bearing of O; that is—

$$\text{angle } SZO = \text{true azimuth of } X + B.$$

If the object be the top of a mountain at a considerable distance, then its true bearing is the same as the Great Circle course from the ship to the latitude and longitude of the mountain's summit, and may be found as directed in the "Treatise on Navigation."

The student should notice that the compass is not used throughout the operation, hence neither its variation nor deviation can affect the result.

Ex. 400. 1887, April 17th, at 3h. 51m. 3s. p.m. mean time at ship in latitude 27° 19′ S., longitude 159° 39′ 45″ E., wishing to swing my ship for a deviation table I found the observed altitude of the sun's U. L. to be 22° 52′ 40″, the height of the eye 17 feet, and the index correction for the sextant − 1′ 17″. At the same instant from the sun's nearest limb towards the west was a point of land in the horizon whose distance was 87° 23′ 50″, measured with a sextant whose index error was + 2′ 13″. Find the true bearing of the point of land.

For Greenwich time.

Mean time at ship, April	17d.	3h.	51m.	3s.
Long. in time, E.	−	10	38	39
G. M. time, April	16	17	12	24

Long. in time.

Long. 159° 39′ 45″
= 10h. 38m. 39s.

For polar distance.

Dec. April 17		10° 27′ 40·7″ N.
Correction	−	5 58·2
True declination		10 21 42·5 N.
S. P. D.		100 21 42·5

Correction for declination.

For 1 hour	−	52·76″
„ 6·79 hours		6·79
		60)358 2404
		5′ 58·2″

For sun's true altitude.

Observed alt. U. L.	22° 52′ 40″
Index error	− 1 17
	22 51 23
Dip	− 4 4
	22 47 19
Semidiameter	− 15 57·6
Apparent altitude	22 31 21·4
Refraction and parallax	− 2 7·3
True altitude	22 29 14·1

For true dist. between sun & land.

Observed distance	87° 23′ 50″
Index error	+ 2 13
	87 26 3
Semidiameter	+ 15 57·6
True distance	87 42 0·6

For sun's true bearing.

(p)	100°	21′	42·5″		
(l)	27	19	0	sec.	·051350
(a)	22	29	14·1	sec.	·034344
	2)150	9	56·6		
S	75	4	58.3	cos.	9·410645
$S \backsim p$	25	16	44·2	cos.	9·956283
					2)19·452622
Half true azimuth	32	10	25·7	sin.	9·726311
∴ True bearing of sun N.	64	21	50·4 W.		

For true bearing of the point of land.

True distance	87°	42′	0·6″	cos.	8 603457
App. alt. of sun	22	31	21·4	sec.	·034456
Diff. of bearings	87	30	36·5 W.	cos.	8·637913
Sun's true bearing, N.	64	20	51·4 W.		
N.	151	51	27·9 W.		
True bearing of point of land, S.	28	8	32·1 W.		

EXERCISE XVI.

Ex. 401. 1887, March 20th, at 7h. 16m. 37s. a.m mean time at ship in latitude 41° 30″ N., longitude 126° 40′ W., the sun bore east by compass. Required the compass error and the deviation for the direction the ship's head had at the time of taking the bearing. Variation 18° 10′ E.

Ex. 402. 1887, November 15th, at 5h. 19m. 30s. p m. mean time at ship in latitude 35° 20′ S., longitude 35° 10′ W., the sun bore by compass west. If the variation was 1° 20′ W., what was the deviation for the direction the ship's head had at the time of observation?

Ex. 403. 1887, May 14th, at 6h. 18m. 36s. a.m. apparent time at ship, the sun's bearing by compass was E. ½ N. Find the deviation for the position of the ship's head at the time if the variation was 25° 40′ E., latitude 52° 20′ N., longitude 129° 10′ W.

Ex. 404. 1887, March 20th, mean time at ship 4h. 11m. 5s. p.m., in latitude 37° 22′ S., longitude 168° 20′ E., the sun bore by compass W. by N. $\frac{3}{4}$ N. If the variation was 13° 20′ E., find the deviation of the compass for the direction the ship's head had at the time.

Ex. 405. 1887, April 14th, mean time at ship 13d. 14h. 34m. 58s. in latitude 35° 20′ S., longitude 40° 28′ E., the variation from the chart was 26° 20′ W. Find the compass error and its deviation for the position of the ship's head if Spica bore by compass N. 70° W.

Ex. 406. 1887, May 11th, at 12h. 10m. 4·5s. mean time at ship in latitude 47° 16′ S., longitude 28° 2′ 30″ W., the observed bearing of the moon's centre by compass was E. by N. $\frac{1}{2}$ N. The variation taken from the chart was 4° 50′ W.; required the error of the compass and the deviation for the direction the ship's head had at the time of taking the bearing.

Ex. 407. If in the English Channel the sun at noon bore S.$\frac{1}{2}$E., what was the deviation for the time and direction of the ship's head if the variation was 19° 20′ W.?

Ex. 408. At the Cape of Good Hope the variation is 30° 10′ W., if a star at its transit over the meridian bears due north by compass. What was the deviation for the then direction of the ship's head?

Ex. 409. 1887, July 27th, mean time at ship 7h. 25m. 3s. a.m. in latitude 32° 30′ N., longitude 64° 36′ W., the observed altitude of the sun's L. L. was 26° 19′ 10″, the height of the eye above the sea was 25 feet, index error − 1′ 6″. At the same instant the distance between the sun's farthest limb and a point of land in the horizon to the northward of the sun was 93° 14′ 10″ with a sextant whose index error was − 37″. What was the true bearing of the point of land?

Ex. 410. Detail a method for obtaining the deviation at sea. How often should this be attended to? Why? Given the variation of the compass, show how its deviation may be obtained at sea. *Royal Naval College*, 1872.

Ex. 411. Prove the rule for finding the true azimuth of the sun, having given the time, the altitude, and declination of the sun, and latitude. How can this be employed to obtain the variation of the compass? *E.* 1871.

Ex. 412. Describe generally the solution of the problem for finding the variation called the "altitude-azimuth" and the "time-azimuth." Draw in each case a figure, and point out

what parts of the triangle of position you have given, and what you have to determine. *A.* 1878.

Ex. 413. Obtain an expression for calculating the sun's true azimuth at a given time and place without observing the altitude. *A.* 1882.

Ex. 414. How does an observation for determining the variation or deviation of the compass differ in practice with regard to the position of an object from what the theoretical considerations would lead you to accept?

Ex. 415. (*a*) Explain with figure and write down the formulæ for a time azimuth. (*b*) Explain why the difficulty in the observation of amplitudes becomes greater in high latitudes.

Royal Naval College, 1869.

Ex. 416. Explain how the true azimuth of a terrestrial object may be determined by (1) astronomical bearings and (2) geographical position. Investigate the problems known as "altitude-azimuth" and the "time-azimuth." *H.* 1880.

Ex. 417. Explain by a diagram what is meant by the altitude and azimuth, north polar distance, and hour angle of a celestial object. Which of these is invariable for a fixed star? and, for two fixed stars, what simple relation connects the two hour angles? *Second B.A. and B.Sc. London*, 1873.

Ex. 418. Describe how the true bearing of a terrestrial object may be found by observing its angular distance from the sun at a known place, having given the ship's mean time of observation. Mention the practical application of the problem, and explain what conditions should be observed that the effects of errors of observation may be diminished to the utmost.

H. 1878.

Ex. 419. Wishing to find the true bearing of a point of land from me with greater accuracy than I could do by means of an azimuth compass, the sun's angular distance from the point of land was observed with a sextant, and at the same time the altitude was taken: show how the true bearing of the point may be computed from these observations.

Royal Naval College, 1864.

CHAPTER XV.

Calculation of altitudes—Proof of formula—Rule—Example—Second method for calculating altitudes—Rule—Example—To find the apparent from the true altitude—Example —Exercise—Examination.

ON THE CALCULATION OF ALTITUDE.

IN clearing lunar distances from the effects of parallax and refraction, in finding the true bearing of a terrestrial object, and in laying down a base line in surveying when the compass cannot be depended on, and in other astronomical problems, a knowledge of the altitudes of objects is often necessary, when, from the obscurity of the horizon by fog, or from the bodies being situated over the land, it is impossible to make observations with accuracy, recourse must then be had to calculating the altitudes. The true altitude of an object is thus found. Beginning with the fundamental formula so often used,—

$$\text{Cos. } z = \cos. p \,.\, \cos. l' + \sin. p \,.\, \sin. l' \,.\, \cos. h$$

$$= \cos. p \,.\, \cos. l' + \sin. p \,.\, \sin. l' \left(2 \cos.^2 \frac{h}{2} - 1\right)$$

$$= \cos. (p + l') + 2 \sin. p \,.\, \sin. l' \cos.^2 \frac{h}{2}$$

$$1 - \cos. z = 1 - \cos. (p + l') - 2 \sin. p \,.\, \sin. l' \cos.^2 \frac{h}{2}$$

$$2 \sin.^2 \frac{z}{2} = 2 \sin.^2 \frac{p + l'}{2} - 2 \sin. p \,.\, \sin. l' \,.\, \cos.^2 \frac{h}{2}.$$

Now, if we assume $\underline{\sin.^2 \theta = \sin. p \,.\, \sin. l' \cos.^2 \frac{h}{2}}$, we get— (1)

$$\sin.^2 \frac{z}{2} = \sin.^2 \frac{p + l'}{2} - \sin.^2 \theta\,;$$

$$\therefore \sin. \frac{z}{2} = \sqrt{\sin. \left(\frac{p + l'}{2} + \theta\right) . \sin. \left(\frac{p + l'}{2} - \theta\right)}. \quad (2)$$

RULE (*a*) Find the Greenwich time, and obtain the polar distance (p).

(*b*) Find the hour angle of the object by rules (p. 93, 94), and divide it by 2. $\left(\frac{h}{2}\right)$.

(*c*) Take half the sum of the polar distance and the colatitude. $\left(\frac{p + l'}{2}\right)$.

(*d*) Calculate θ thus :—To twice log. cosine of half the hour angle, add log. sine of colatitude and log. sine of the polar distance; half their sum is log. sine θ.

$$\sin. \theta = \sqrt{\sin. p \,.\, \sin. l' \,.\, \cos.^2 \frac{h}{2}}.$$

(*e*) Take the sum and difference of (*c*) and (*d*) $\left(\frac{p + l'}{2} + \theta\right)$ and $\left(\frac{p + l'}{2} - \theta\right)$, and add log. sines of these quantities together; half their sum is log. sine of half the zenith distance.

$$\sin. \frac{z}{2} = \sqrt{\sin. \left(\frac{p + l'}{2} + \theta\right) . \sin. \left(\frac{p + l'}{2} - \theta\right)}.$$

(*f*) The double of half the zenith distance subtracted from 90° gives the true altitude.

Ex. 420. 1887, November 20th, at 3h. 13m. 40s. mean time at ship in latitude 41° 56′ S., longitude 38° 49′ E., find the sun's true altitude.

For Greenwich time.

Mean time ship, Nov. 20	3h. 13m. 40s.
Long., E.	− 2 35 16
Greenwich M. T. Nov. 20	0 38 24

Long. in time.

Long. 38° 49′ E.
= 2h. 35m. 16s.

For sun's polar distance.

Declination Nov. 20	19° 41′ 57″ S.
Correction	+ 21·8
True declination	19 42 18.8 S.
S. P. D.	70 17 41·2

Cor. for declination.

For 1h.	+ 34·08″
„ 38m. 24s.	·64
Correction	21·8112

For equation of time.

November 20	14m. 15·04s.
Correction	− ·39
True E. T. +	14 14·65

Correction for E. T.

For 1 hour	− ·603s.
	·64
Correction	·38592

For sun's hour angle.

Mean time ship	3h.	13m.	40s.
Equation of time	+	14	14·65
App. time ship	3	27	54·65
	60		
	4)207		54·65
Sun's hour angle	51°	58′	40″
$\therefore \frac{h}{2}$	25	59	20

For sun's altitude.

$\frac{h}{2}$	= 25°	59′	20″	cos.	9·953701
					2
				cos.^{2}	19·907402
l'	= 48	4	0	sin.	9·871528
p	= 70	17	41·2	sin.	9·973793
$\frac{p+l'}{2}$	= 59	10	50·6		2)19·752723
θ	= 48	47	7·8	sin.	9·876361
Sum	107	57	58·4	sin.	9·978289
Diff.	10	23	42·8	sin.	9·256325
					2)19·234614
$\frac{z}{2}$	24	28	29·2	sin.	9·617307

Zenith distance	48°	56′	58·4″	
True altitude	41	3	1·6	*Answer.*

Another method for finding the true altitude of a celestial object is as follows, and possesses the merit of greater simplicity in its application :—

Let the celestial concave be projected on the plane of the horizon, X as before being the position of the object. From

Fig 43

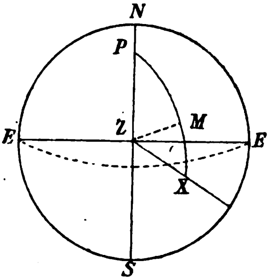
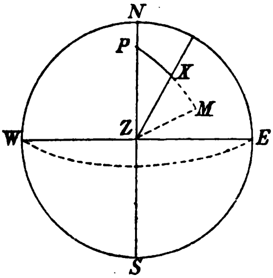


Fig 44



Z draw $Z M$ perpendicular to $P X$ or to $P X$ produced. Then in the right-angled triangle $P Z M$

$$\cos. h = \tan. P M . \cot. l' ;$$
$$\therefore \tan. P M = \cos. h . \tan. l' ;$$

and if $P M = \theta$, $\tan. \theta = \cos. h . \cot. l$ . . . . I.

From the figures $M X = P X \backsim P M$

$$= (\theta \backsim p) \quad . \ . \ . \ . \quad (a)$$

also $$\cos. Z M = \cos. P Z . \sec. P M$$
$$= \sin. l . \sec. \theta \quad . \ . \quad (b).$$

In the triangle $Z M X$

$$\cos. Z X = \cos. Z M . \cos. M X.$$

From (a) and (b)

$$\therefore \sin. \text{alt.} = \sin. l . \sec. \theta . \cos. (\theta \backsim p) \quad . \ . \text{ II.}$$

Equations I. and II. are all that are used in the solution of the problem, and may be deduced from the fundamental formula thus:—

$$\text{Cos. } z = \cos. p . \cos. l' + \sin. p . \sin. l' . \cos. h$$
$$= \sin. l \{\cos. p + \sin. p . \cot. l . \cos. h\}.$$

Now if $\underline{\tan. \theta = \cot. l . \cos. h}$ . . . . I.,

then $$\cos. z = \sin. l (\cos. p + \sin. p . \tan. \theta)$$
$$= \sin. l \frac{\cos p . \cos. \theta + \sin. p . \sin. \theta}{\cos. \theta} ;$$
$$\underline{\therefore \sin. \text{alt.} = \sin. l . \sec. \theta . \cos. (\theta \backsim p)} \quad . \ . \ . \ . \ . \text{ II.,}$$

thus agreeing with the fotmulæ obtained direct from the figures.

In formula I, *cot.* l is always positive, but h may be less or greater than 90°, or six hours. If less, cot. θ is positive, and the angle taken from the tables will be the value of θ; but if

h be greater than six hours, cot. θ will be negative: then the supplement of the angle taken from the tables gives the value of θ. From these formulæ we deduce the following:—

Rule (*a*) Get the Greenwich time and find the polar distance (p).

(*b*) Find the hour angle by rules, p. 93, 94, (h).

(*c*) To log. cos. h, add log. cot. lat.; the result is log. tan. θ if the hour angle be less than six hours; but if the hour angle be greater than six hours the angle taken from the tables must be subtracted from 180° for θ.

(*d*) Take the difference between θ and the polar distance, and add together log. sin. lat., log. sec. θ. and log. cos. ($\theta \backsim p$); the result is log. sine of the altitude.

Ex. 421. 1887, November 20th, at 3h. 13m. 40s. mean time at ship in latitude 41° 56′ S., longitude 38° 49′ E., find the sun's true altitude.

This question is the same as worked by the previous method, and we may, therefore, use the data there found: then

h	51° 58′ 40″	cos. 9·789557			
l	41 56 0	cot. ·046579	sin. 9·824949		
θ	34 26 18·8	tan. 9·836136	sec. ·083686		
p	70 17 41·2				
$\theta \backsim p$	35 51 22·4	. . . .	cos. 9·908747		
True altitude		41° 3′ 1·2″	sin. 9·817382		

It will be seen by reference this agrees with the answer previously found.

On finding the apparent altitude when the true is known.—It will be evident that if the true altitude be known and the apparent be required from it, we must proceed in the reverse order to that by which the true altitude was found from the apparent; that is, *subtract* the parallax and *add* the refraction. This will be required in every instance in which the altitudes are computed for the purpose of clearing a lunar distance. It is done as follows:—

(A) For a Star.—Add the refraction to the true altitude. If the altitude be low, first perform the process mentally, then, using the altitude thus found, take out

the refraction and add. As refraction is great for objects near the horizon, care must be exercised in such instances in applying the proper refraction.

Ex. 422. Find the apparent altitude of Rigel if its true be 14° 11′ 45″.

True altitude	14° 11′ 45″
Refraction	+ 3 41·5
Apparent altitude	14 15 26·5

By mentally adding the refraction for 14° 11′ 45″, viz. 3′ 41″, the apparent altitude of the star is found to be about 14° 15′ 28″: then, taking the refraction for this quantity, we get 3′ 41·5″ to be added, which differs from the former correction by 1·5″.

(B) FOR THE SUN.—(*a*) Subtract the parallax in altitude.

(*b*) Take out the refraction and add, but use the same precaution when dealing with every other object when the altitude is low, as with a star.

Ex. 423. The true altitude of the sun's centre is 45° 17′ 21″. Find the apparent altitude.

True altitude	45° 17′ 21″
Parallax	− 6
	45 17 15
Refraction	+ 56·4
Apparent altitude	45 18 11·4

(C) FOR THE MOON.—(*a*) Find the horizontal parallax for a given time.

(*b*) Calculate the parallax in altitude from the formula—

par. in alt. = hor. par. × *cos. app. alt.*

(*c*) Subtract this parallax in altitude.

(*d*) Recompute the parallax in altitude with the apparent altitude found in (*c*), and subtract this from the altitude corrected for refraction instead of that found in *b*.

Ex. 424. Find the apparent altitude of the moon's centre when her horizontal parallax is 60′ 27″, and her true altitude 37° 18′ 44″.

Par. in alt. = *hor. par.* × *cos. app. alt.*

True alt.	37°	18′	44″		
Refraction	+	1	17		
	37	20	1	cos.	9·900431
Hor. par.		36	27	log.	3·559548*
Par. in alt. nearly		2884		log.	3·459979
	=	48′	4″		
Altitude	37	20	1		3·559548*
App. alt. nearly	36	31	57	cos.	9·904997
		2914·3″		log.	3·464545
True par. in alt.		48′	34·3		
	37	20	1		
Apparent altitude	36	31	26·7		

(D) FOR A PLANET.—This process is precisely the same as for the moon, except that as the parallax in altitude is so small no recomputation need be made.

Ex. 425. The true altitude of Venus was 60° 24′ 25″. Find the apparent altitude when her horizontal parallax was 25″.

	Hor. par.		25″	log.	1·397940
True altitude	60°	24′	25″	cos. log.	9·693580
Par. in alt.		−	12·3	log.	1·091520
	60	24	12·7		
Refraction		+	32·6		
Apparent altitude	60	24	45·3		

Correction of altitude for a small elapsed time.—In making observations at sea it is often convenient to reduce the altitude of an object to what it would be a few minutes later. This is necessary when only one observer takes the whole of the observations for a lunar distance, and is performed by first taking the altitude of the object, then the distance between it and the moon, and, finally, the altitude again of the same object, noticing the exact moment of each observation. Then, assuming the change in altitude to be uniform, the altitude at any intermediate instant is calculated by proportion. If the interval be small this will lead to no appreciable error, and the method used will be easily understood from the following example

Ex. 426. At 5h. 30m. 25s. Jupiter's altitude was 32° 2′ 10″; the distance between the moon and Jupiter was taken at 5h. 32m. 40s.; and Jupiter's altitude at 5h. 34m. 10s. was 32° 3′ 30″. Required the altitude of the planet at the time of taking the distance.

Time elapsed between taking 1st alt. and distance = 2m. 15s.
" " " " 2nd. alt. = 3 45
Change of altitude in time = + 1′20″

The question then resolves itself into: If the change in altitude be 1′ 20″ in 3m. 45s., what is the change in 2m. 15s.? On working this out it is found to be 48″. Then

Altitude at 5h.	30m.	25s.	=	32° 2′ 10″
Change in	2	15	=	+ 48
∴ Altitude at 5	32	40	=	32 2 58

Ex. 427. 1887, February 13th, at 8h. 17m. 16s. a.m. mean time at ship in latitude 12° 18′ N., longitude 126° 47′ W., find the observed altitude of the moon's lower limb, if the height of the eye be 19 feet, and index error of the sextant + 3′ 5″.

For Greenwich date.

Mean time ship, Feb.	12d.	20h.	17m.	16s.
Long., W.		+ 8	27	8
Greenwich M. T., Feb.	13	4	44	24

Long. in time.

Long. 126° 47′ W.
= 8h. 27m 8s.

For moon's semidiameter.

Noon, February 13	16′ 8·5″
Correction	+ 2·6
	16 5·9
Augmentation	+ 9·5
Correct S. D.	16 15·4

Variation.

In 12 hours	− 6·6″
No. hours	4·74
	12)31·284
Cor. for 4·74 hrs.	− 2·6

For moon's horizontal parallax.

Noon, February 13	59′ 8·4″
Correction	− 9·6
	58 58·8
Reduction	− ·7
Correct H. P.	58 58·1 = 3538·1″

Variation.

In 12 hours	− 24·2″
No. hours	4·74
	12)114·708
Cor. for 4·74 hrs.	− 9·559

For moon's R. A.

February 13th, 4h.	14h.	24m.	24·72s.
Correction	+	1	40 58
True R. A. moon	14	26	5 30

Correction.

For 10m.	+ 22·653
	4·44
	100·57932

For moon's polar distance.

February, 13d. 4h.	9°	7′	8·9″ S.
Correction	+	7	32·8
True declination of moon	9	14	41·7 S.
∴ Moon's N. P. D.	99	14	41·7

Correction.

For 10m. + 101·98
4·44
60)452·7913
7′ 32·8″

For R. A. mean sun.

Sidereal time, July 13, 21h.	32m.	48·97s.
Acceleration for 4h.		39·43
Acceleration for 44m.		7·23
Acceleration for 24s.		·06
R. A. mean sun 21	33	35·69

For moon's hour angle.

R. A. mean sun	21h.	33m.	35·69s.
M. T. place	20	17	16
R. A. meridian	17	50	51·69
R. A. moon	14	26	5·30
Moon's h. angle	3	24	46·39

Western hour angle 51° 11′ 36″

$\therefore \frac{h}{2} = 25° \ 35' \ 48''$

For moon's true altitude.

$\left(\frac{h}{2}\right)$	25°	35′	48″	cos.	9·955138
					2
				cos.²	19·910276
(l')	77	42	0	sin.	9·989915
(p)	99	14	41·7	sin.	9·994322
$\frac{p+l'}{2}$	88	28	20·85		2)19·894513
θ	62	19	48·64	sin.	9·947256
Sum	150	48	9·49	sin.	9·688259
Diff.	26	8	32·21	sin.	9·644046
					2)19·332305
$\frac{z}{2}$	27°	37′	13″	sin.	9·666153
Moon's zen. dist.	55	14	26		
Moon's true alt.	34	45	34		
Hor. par.	—	58	58		
	33	46	36	Hor. par. 3538·1″ log.	3·548770
Refraction	+	1	24		
App. alt. nearly	33	48	0	. . . . . . cos.	9·919593
∴ Par. in alt.		49	0	2940″ log.	3·468363

By recomputing the parallax in altitude it will be found to be 48′ 55″.

For moon's observed altitude.

Moon's true altitude	34° 45′ 34″
Par. in alt.	− 48 55
	33 56 39
Refraction	+ 1 24·9
Moon's apparent altitude	33 58 3·9
Semidiameter	− 16 15·4
	33 41 48·5
Dip for 19 feet	+ 4 17
	33 46 5·5
Index error	− 3 5
Observed altitude	33 43 0·5

The true altitude is calculated by the second method thus :—

h	51° 11′ 36″	cos.	9·797056		
l	12 18 0	cot.	661473	sin.	9·328442
θ	70 48 59·6	tan.	10·458529	sec.	·483341
p	99 14 41·7				
$\theta \backsim p$	28 25 42·1	. . . . .		cos.	9·944193
True altitude	34° 45′ 34″			sin.	9·755976

EXERCISE XVII.

Ex. 428. If the altitude of α Serpentis change from 46° 18′ 25″ to 45° 48′ 40″ in 10m. 25s., what is its altitude at 5m. 5s. from the time of the first observation ?

Ex. 429. What were the altitudes of the moon and Antares at the time of taking the annexed distance ?

10h.	27m.	35s.	alt. of Antares	29° 18′ 35″
10	29	5	„ moon	37 45 15
10	30	50	dist. star from moon	87 16 18
10	32	0	alt. of moon	38 15 25
10	33	25	„ Antares	30 10 25

Ex. 430. Find the altitudes of the moon and Regulus at the time of taking the distance between them.

3h.	20m.	10s.	alt. of Regulus	45°	18′	50″
3	22	40	„ moon	39	27	30
3	25	20	dist. star from moon	69	2	48
3	27	45	alt. of moon	38	30	45
3	30	15	„ Regulus	46	3	20

Ex. 431. From the following observations reduce the altitudes to the time of taking the distance:—

18h.	10m.	45s.	alt. of α Arietes	20°	18′	44″
18	13	15	„ moon	37	38	20
18	16	15	dist. star from moon	49	51	11
18	18	45	alt. of moon	38	30	10
18	21	15	„ α Arietes	19	27	20

Ex. 432. 1887, February 13th, in latitude 12° 18′ N., longitude 126° 47′ W., find the true and apparent altitudes of the sun at 8h. 17m. 16s. a.m. mean time at place.

Ex. 433. 1887, August 14th, find the true and apparent altitudes of the sun in latitude 27° 20′ N., longitude 93° 20′ 45″ W., at 10h. 40m. a.m. mean time at place.

Ex. 434. 1887, October 3rd, at 4h. 37m. a.m. mean time at ship, find the true and apparent altitudes of Aldebaran in latitude 14° 39′ S., longitude 42° 51′ E.

Ex. 435. 1887, September 3rd, mean time at Greenwich 4h. 53m. 22s. p.m. in latitude 27° 45′ S., longitude 166° 20′ 15″ E.; find the true and apparent altitudes of Fomalhaut.

Ex. 436. 1887, May 24th, in latitude 33° 36′ N., longitude 30° 50′ E., at 3h. 20m. 44s. p.m. mean time, find the true and apparent altitudes of the moon.

Ex. 437. 1887, March 17th, at 3h. 23m. 48s. a.m. mean time at ship, in latitude 33° 46′ N., longitude 72° 21′ W., find the true and apparent altitudes of the moon.

Ex. 438. 1887, July 7th. Find the true and apparent altitudes of Jupiter at 9h. 22m. p.m. mean time at ship in latitude 48° 23′ S., longitude 164° 23′ 30″ W.

Ex. 439. 1887, April 5th, at 10h. 27m. 43s. a.m. mean time at Greenwich, find the true and apparent altitudes of Venus in latitude 18° 42′ N., longitude 136° 20′ E.

Ex. 440. 1887, February 8th, mean time at ship 2h. 48m. 20s. a.m. in latitude 19° 52′ S., longitude 141° 16′ 15″ E.;

find the observed altitude of Pollux, if the index error of the sextant was + 3′ 12″, and height of the eye 18 feet.

Ex. 441. 1887, December 31st, 2h. 39m. 4s. a.m. mean time at ship in latitude 39° 53′ S., longitude 11° 38′ 30″ E.; find the observed altitude of the moon's upper limb if the height of the eye above the sea was 25 feet, and the index correction for the sextant was + 2′ 15″.

Ex. 442. Prove the rule for computing the altitude of a heavenly body for a given time at a given place. What is the true altitude of the sun's L. L. on June 18th, 1887, at a place in latitude 47° 50′ N., and longitude 14° 30′ E., at 10 a.m. app. time? Find also what altitude would be observed, given the height of the eye 18 feet, the index correction — 3′ 40″. *A.* 1869.

Ex. 443. Describe fully how an altitude taken at one instant may be reduced to what it would be at any other near instant.

Ex. 444. In finding the longitude by lunar observation, describe the method adopted when the altitudes cannot be obtained at the same time as the distance is taken on account of the obscurity of the horizon.

As an example, calculate the apparent altitudes of the sun and moon with the following data: latitude 28° 30′ N., sun's declination 20° 8′ 44″ N.; moon's declination, 18° 50′ 40″ N.; moon's horizontal parallax 54′ 49″; sun's hour angle, 22h. 30m. 10s.; moon's hour angle, 4h. 23m. 44s. *H.* 1878.

Ex. 445. In latitude 40° 48′ N., at 4h. 35m. ship's apparent time, an object *O* in the horizon was observed to be distant 70° 14′ 30″ from the sun's nearest limb: *O* was to the right of the sun and W. of the meridian. Find the true bearing of *O*, having given

Sun's semidiameter, 16′ 2″
Sun's declination, 4° 35′ N.

For Lieutenant, 1874.

CHAPTER XVI.

Double altitudes—Advantages of data employed—How the polar angle is found—Correction for run—Proof of formula—Exercise—The direct method of double altitudes—Proof of formula—Rule—Example—Exercise—Ivory's method of double altitudes—Proof of formula—Rule—Example—Corrections necessary for latitude and longitude found by Ivory's method—Proof of formula—Example—Exercise—Sumner's method of double altitudes—Circles and lines of position—Proof of rule—Rule—Example—Projection of the example—Advantages of Sumner's method—Calculation of direction of line of position and thence the error of the compass—Exercise—Short method of double altitudes—Proof of formula—Rule—Example—Exercise—Examination.

Double altitudes, as the name implies, is the method used for determining the ship's position by means of two altitudes or two sets of altitudes of the same or different bodies. When two bodies are observed at the same time, two observers must perform the operation: and in all cases the time must also be noted. When two bodies are observed at different times by the same person, the observations may be made as follows. The altitude of the object nearest the meridian must be first observed, because its motion in altitude is more uniform than that of one more remote, then the altitude of the second object, and lastly, the altitude again of the first observed object, and the times of each observation must be noted. Then by simple proportion the altitude of the first object must be reduced to what it would be at the time of taking the altitude of the second. When the same object (as the sun) is observed, some time must elapse between taking the altitudes, as shown hereafter.

ADVANTAGES OF DOUBLE ALTITUDES.—From the remarks which have been made when treating of "ex-meridian altitudes," it is evident that that method of finding the latitude cannot be adopted if there be any uncertainty in the time at ship. To remove this difficulty recourse is had to double altitudes. In the application of this problem, *absolute time* is never required, but only *the interval of time* elapsed between the two sets of

observations must be known with accuracy, and this is easily obtained from the chronometer by applying the proportional part of its daily rate to the observed interval. Because latitude is deduced easier from a meridian altitude than by any other method, recourse is only had to "reduction to the meridian" and to "double altitudes" when the state of the weather precludes observations on the meridian. Many methods for the solution of the problem of double altitudes have been proposed by different writers, but we shall confine ourselves to the direct method, Ivory's method, Sumner's method, and to the short method.

DATA EMPLOYED.—The data used in the calculations are:—

(1) The *polar angle*, or the angle included between circles of declination which pass through the points in the heavens which were occupied by the objects when the altitudes were taken.

(2) The *polar distances*, which are obtained by means of the declinations corrected for the Greenwich time of observations; and

(3) The *zenith distances*, which are deduced from the observed altitudes.

The *polar angle* is the only element likely to give any trouble to the learner, and as it is deduced differently for different objects, we shall discuss it under the following heads:—

(*a*) When altitudes of the sun only are used.

(*b*) When altitudes of the same star are used.

(*c*) When altitudes of two stars are taken by two observers at the same time.

(*d*) When altitudes of two stars are taken in quick succession by the same observer.

(*a*) *When altitudes of the sun only are used.*—After taking one set of altitudes and noting the exact instant by a watch or chronometer whose rate is known, sufficient time should be allowed to elapse for the sun to change his azimuth from 60° to 120°, and another set of altitudes should be taken and the time again noted. To the time elapsed between the two sets, the correction for the rate of the timekeeper used must be applied; this will give the precise interval in *mean* solar time; but as the polar angle, in this case, is the elapsed *apparent* time, the mean interval must be converted into an apparent interval by applying a proportional part of the equation of time for the interval elapsed between the observations. The rule for applying the correction for the equation of time is (see pp. 80-1):—

If it be *additive* to mean time and *increasing*, }
or if it be *subtractive* from „ *decreasing* } add.
But
If it be *additive* to mean time and *decreasing*, }
or if it be *subtractive* from „ *increasing* } subtract
from the elapsed mean interval to obtain the polar angle.

The reasons for these maxims have been already given, pp. 80-1, but in practice these corrections are so minute that they may safely be disregarded.

(*b*) *When altitudes of the same star are used.*—The difference in azimuth should in this and, in fact, in every case in double altitudes, lie between 60° and 120° for reasons hereafter to be proved. The polar angle is the elapsed sidereal time: hence the mean solar interval, deduced as in (*a*), must be turned into a sidereal interval by the "table of time equivalents" in the "Nautical Almanac" or by the decimal multiplier.

(*c*) *When altitudes of two stars are taken by two observers at the same instant.*—The best time for these altitudes is at twilight, because then both stars and horizon can be seen. The angle included between their declination circles, i.e. the difference in their right ascensions, is the polar angle. This will be found to be the simplest case which can occur, as no corrections are necessary for the error of the watch, nor for the run of the ship during the interval, and thus all errors of observation for run and for imperfections of compasses are avoided. No absolute time of taking the observations is required, because the stars do not change their declinations; there is no fear of losing a second observation, and the polar angle is very easily calculated.

(*d*) *When altitudes of two stars are taken in quick succession by the same observer.*—A short interval must elapse between the observations; this must be turned into a sidereal interval: and because the stars appear to move westward through the sky, this sidereal interval must be added to the right ascension of the star first observed. Then, as in (*c*), the difference between the corrected right ascension of the star first observed, and that of the other, will give the polar angle. Or, take the altitude of the star nearest the meridian (because its change of altitude is more uniform), note the time by watch; then take the altitude of the other star, noting the time; and lastly, take the altitude of the star first observed again, once more noting the time. From the change of altitude of the first star in the elapsed

time between observing it, its altitude at the instant when the second star was observed may be found by simple proportion. Then this case is reduced to that of the last.

CORRECTION FOR RUN.—When one object alone is observed, a correction is required to one of the zenith distances, because the ship seldom, if ever, remains in the same place during the interval between taking the two altitudes: hence, if the method of double altitudes by one object is to be of any practical utility, the observations at one station must be reduced, when necessary, to what they would have been at the same time at the other. This is called the *correction for run.* The first to show the necessity for this correction was Mr. Nicholas Facio Duillier, F.R.S., in a pamphlet called "Navigation Improved," published by him in 1728; but Graham, in the "Philosophical Transactions" for the year 1734, No. 435, seems to have been the first who presented the problem in its present form.

Let $M\,N$ be a meridian, Z the zenith at the first observation, then as the ship alters her position, the zenith of the observer must describe a similar path in the sky. Let Z' be the zenith at the second observation, then the arc $Z\,Z'$ must measure the distance run between the observations, which never exceeds a few miles. Let X be the position of the heavenly body at the first observation, which is supposed fixed while the observer's zenith moves from Z to Z'. With the centre Z and distance $Z\,Z'$ describe an arc of a small circle $Z'\,V$, cutting $X\,Z$ or $X\,Z$ produced in V. Then $Z\,V$ is the distance the zenith has moved towards or from the object X during the interval; hence this will be the difference in altitudes, supposing the observations to be made simultaneously at the two stations, and is therefore the correction to be applied to the altitude first observed for the run of the ship in the interval.

Fig 45

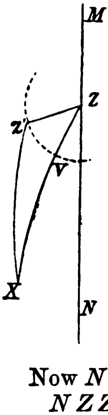
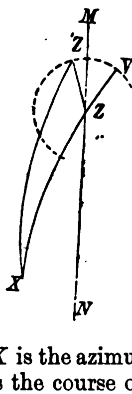


Fig 46



Now $N\,Z\,X$ is the azimuth of the body at the first observation, $N\,Z\,Z'$ is the course of the ship during the interval,

$\therefore\ X\,Z\,Z' = N\,Z\,Z' - N\,Z\,X =$ the difference between the course of the ship and the bearing of the object.

Again, the triangle $Z Z' V$ being so small, no material error will be introduced by considering it a plane triangle right-angled at V.

Then $Z V$ or correction for run $= Z Z' . \cos. Z' Z X =$ distance run $\times$ cos. angle between the course during the interval and the bearing of the object.

It is very evident, if the run has been towards the object, the altitude of the body at the first station would have been greater if taken at the same time at the second: this is shown in Fig. 45, where the zenith distance $Z X$ is too great for an observation at the second station by the arc $Z V$; or, what amounts to the same thing, the altitude is too little by the arc $Z V$. Hence if the angle between the course of the ship and the bearing of the object be *less than* eight points or 90°, the correction, as found above, must be *added* to the altitude at the first observation to reduce it to what it would have been at the second.

If the run has been away from the object, the altitude at the first station would have been less if taken at the same time at the second, as shown in Fig. 46, where the zenith distance $Z X$ is too little by the arc $Z V$—i.e. the altitude is too great by the same arc. Hence if the angle between the course of the ship and the bearing of the object be *greater than* eight points or 90°, the correction must be *subtracted* from the altitude at the first observation.

If the run has been directly towards the object, the whole distance made good must be added; if the run has been directly away from the object, the whole distance made good must be subtracted from the first observation; and if the angle between the course of the ship and the bearing of the object be exactly eight points, there will be no correction for run, because $\cos. 90° = 0$. From these considerations the following rule is derived:—

Rule.—The traverse table supplies a ready means for correcting for run, thus:—

(1) Find the elapsed time between the observations and the distance made good in the interval.

(2) Find the angle between the course made good by the ship during the interval and the bearing of the object.

(3) Turn to the traverse table. With the angle found in (1) as a course, and the run as distance, the difference of latitude will be the correction for run, to be applied according to the precepts above. This follows directly from *diff. lat. = dist.* $\times$ *cos. course.*

The course of the ship must be corrected for leeway, &c.; and if more than one has been sailed on during the interval, these must all be reduced to one resultant course and distance before finding the angle between the course made good and the bearing of the object.

Ex. 446. An altitude of the sun was taken at 8h. 37m. 50s. a.m., and his bearing at the instant was S. 83° 50′ E. At 0h. 50m. 38s. the altitude was taken again. Required the correction to be applied to the first observation if the vessel had made good a S. by E. course, $7\frac{1}{2}$ knots per hour, during the interval.

For elapsed time.	*For run.*	*For included angle.*
12h. 50m. 38s.	Rate 7·5m.	Sun's bearing, S. 83° 50′ E.
8 37 50	Interval 4·2	Ship's course, S. 11 15 E.
4 12 48	150	72 35
	300	
	31·50m.	

Included angle	72° 35′	cos. =	9·476133
Run	31·5	log. =	1·498311
Correction	9·429	log. =	0·974444
	60		
	25·740		

∴ Correction for run = + 9′ 25·7″.

By looking in the traverse table for the course $72\frac{1}{2}°$ and distance 31·5 miles, we have in the difference of latitude column 9·5 miles = 9′ 30″, which differs from the correct amount above by 4·3″ only. And as the angle between the sun's bearing and the ship's course is less than eight points, this correction must be added to the first altitude.

Now, as the interval between the observations is always small, never more than five or six hours, the polar angle may be considered to be correct; but at sea the correction for run must necessarily be an approximation, because it involves the bearing of the object and the ship's course and distance during the interval. The former two may be in error, from not knowing

the exact leeway the vessel has been making, nor the correct variation nor deviation for her geographical position, or from bad steering or imperfections in the compass itself. The distance may be in error from not knowing the exact rate of the vessel, nor the set and rate of the current through which she has been passing during the interval, &c. If the true bearing of an object could be found with exactitude, double altitudes and other ex-meridian altitudes would fall into disuse, for we should then have the polar distance of an object, its zenith distance, and its azimuth to find the colatitude, i.e. two sides and an angle opposite to one of them to find the third side and the other angles, well known cases in spherical trigonometry. From these considerations an element of uncertainty is always introduced into one of the altitudes. This uncertainty may be reduced to a minimum by attending to the following maxims :—

(1) When observations are made on the same side of the meridian, the difference of azimuth at the two observations should exceed the less azimuth.

(2) When on different sides of the meridian, the supplement of the difference of azimuth at the two observations should exceed the less azimuth.

(3) This method should not be used when a great circle, joining the positions of the object or objects, passes within two or three degrees of either the zenith or the pole.

(4) The distance run by the ship in the interval should not exceed 36 to 40 miles.

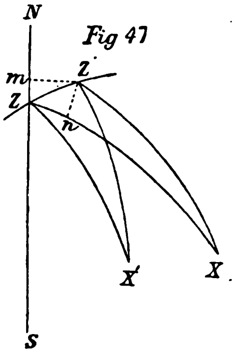


Fig 47

These maxims are deduced from the following considerations :—

ON THE MOST ADVANTAGEOUS POSITION FOR OBSERVATION.—Let $N S$ be a meridian, X and X' the positions of the sun at the two observations, Z the zenith used in calculating the latitude, and let Z' be the true zenith; then $Z X' = Z' X'$ if the altitude when the object has the least azimuth be considered the correct one; but $Z' X$ differs from $Z X$ by a small quantity equal to the error in altitude. From Z' draw $Z' n$ and $Z' m$ perpendicularly to $Z X$ and to the meridian respectively, then $Z m$ is the error in latitude corresponding to $Z n$ the error in altitude.

Draw $Z\,Z'$ an arc of a circle around the pole X'.

Now, angle $m\,Z\,Z' + Z'\,Z\,X' + X'\,Z\,S =$ two right angles, and $Z'\,Z\,X$ is a right angle;

$\therefore\; m\,Z\,Z'$ is complementary to $X'\,Z\,S$ . . . (a),

and $\because\; X'\,Z\,Z'$ is a right angle.

$Z'\,Z\,n$ is complementary to $X\,Z\,X'$ . . . . (b).

Again, because the distance run is so small, the triangles $Z'\,m\,Z$ and $Z'\,n\,Z$ may be considered as plane ones.

Then—

$$Z\,m = Z\,Z' \,.\, \cos.\, m\,Z\,Z'$$

$$= Z\,n\,\frac{\cos.\, m\,Z\,Z'}{\cos.\, n\,Z\,Z'}$$

$$= Z\,n\,\frac{\sin.\, X'\,Z\,S}{\sin.\, X'\,Z\,X} \text{ from } (a) \text{ and } (b),$$

that is, error in lat. at 2nd observation

$$= \text{error in alt.} \times \frac{\text{sin. az. at 2nd observation}}{\text{sine diff. of bearings}} \quad . \quad (c);$$

hence, for observations on the same side of the meridian, the error in latitude will be least for an error in altitude when the numerator of the fraction in (c) is least and the denominator the greatest possible; that is, when the object at the least azimuth is near the meridian, and the difference of bearings as near 90° as possible; and for observations on different sides of the meridian the difference of bearings should be as small as possible. The bearing of the object should be always taken at the less altitude, because it can then be done with greater precision. If the less altitude for the same object be taken last, the ship's course in the interval should be reversed, and the correction for run computed as before; the reversed course must be used to ascertain if the run has been towards or from the object. The less altitude should be always the one to be corrected for run in practice, whilst the latitude should be found for the greater altitude, which should be taken as near noon as possible.

EXERCISE XVIII.

Ex. 447. In a double altitude what is the meaning of the correction for run? Find the correction in the following cases:—

First bearing, N. . . . . . Run 10′ E.
" " N. . . . . . Run 10′ S.W.½W.
" " N. . . . . . Run 10′ S.

For Lieutenant, 1873.

Ex. 448. An altitude of the sun was taken at 11h. 43m. 20s. a.m., and at 4h. 17m. 10s. p.m. his altitude was again taken. If the magnetic bearing at the first observation was S. by E. $\frac{3}{4}$ E., and the course and distance made good in the interval E. by S. $\frac{1}{2}$ S., $5\frac{3}{4}$ knots per hour, find the correction for run to be applied to the first altitude.

Ex. 449. 1887, May 23rd, an observation of the sun was taken by chronometer at 7h. 43m. 51s. a.m., and at 0h. 13m. 7s. p.m., by the same chronometer, another observation was made; find the angle at the pole between the hour circles through the sun at the times of observation, *the rate being disregarded.*

Ex. 450. Show how the polar angle is found in every case that can occur in finding latitude by double altitudes.

Ex. 451. Prove the rule for applying the equation of time in converting a mean solar interval into an apparent solar interval.

Ex. 452. When the same observer uses two stars for finding latitude, with a short interval between the observations, show how to reduce the altitude of one star to what it would have been if taken at the same instant as the other.

Ex. 453. Why is the angle between the direction of the ship's run and the bearing of the sun at the first observation involved in the correction for run?

Royal Naval College, 1872.

Ex. 454. Explain clearly the nature of the following corrections:—Index correction—Parallax—Run. Prove that

$$\text{Correction for run} = \text{dist. run} \times \text{cos. angle of run.}$$

State what is meant here by "angle of run;" and explain why the traverse table can be used to determine the correction for run. *A.* 1876.

Ex. 455. In determining the latitude at sea by means of two non-meridian altitudes of the sun, taken at two different times of the day, investigate by any method the correction to be applied for the ship's change of place during the interval between the observations. *Second B.Sc. London*, 1878.

Ex. 456. Show where objects should be situate in deducing latitude from double altitudes of heavenly bodies, so that there shall be least chance of error in the result

(*a*) When the bodies are on *the same* side of the meridian.

(*b*) When the bodies are on *different* sides of the meridian.

THE DIRECT METHOD OF DOUBLE ALTITUDES.

This is sometimes called *Delambre's method*, and, as its name implies, the problem is solved directly by formulæ proved in all works on spherical trigonometry. It is universally applicable; and Captain Kater says,[1] when comparing Ivory's method with it, "But as this requires as many references to tables, and a correction on account of the change in the declination of the sun, we have been induced to prefer the direct process."

Fig 48

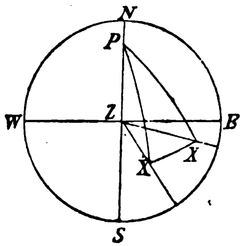
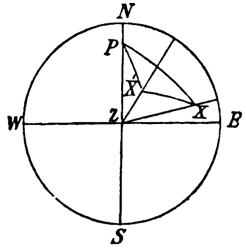


Fig 49



PROOF.—We will now proceed to show how the formulæ from spherical trigonometry is applied in the solution of the problem. Let $N E S W$ be the plane of the horizon, P the elevated pole, Z the zenith. Let X be the object at the greatest distance from the meridian, X' the object nearer the meridian. Join $P X$, $P X'$, $Z X$, $Z X'$, and $X X'$ by arcs of great circles; then $P\ X = p =$ polar distance of X, $Z X = z =$ zenith distance of X, $P X' = p' =$ polar distance of X', $Z X' = z' =$ zenith distance of X', $X X' = D =$ distance between the objects and $X P X' = P =$ the polar angle.

First. In the triangle $P X X'$ are given $P X$, $P X'$, and the angle $X P X'$, i.e. two sides and the included angle to find D, the third side. This may be effected by any of the usual formulæ for that purpose; but the following is selected as being simple and free from ambiguity:—

$$\begin{aligned}\text{Cos. } D &= \cos. p \,.\, \cos. p' + \sin. p \,.\, \sin. p' \,.\, \cos. P \\ &= \cos. p \,.\, \cos. p' + \sin. p \,.\, \sin. p' \left(2 \cos.^2 \frac{P}{2} - 1\right) \\ &= \cos. (p + p') + 2 \sin. p \,.\, \sin. p' \cos.^2 \frac{P}{2}.\end{aligned}$$

[1] "Encyclopædia Metropolitana," p. 618.

Subtract each side from 1—

$$1 - \cos. D = 1 - \cos. (p + p') - 2 \sin. p \,.\, \sin. p' \,.\, \cos.^2 \frac{P}{2}$$

$$2 \sin.^2 \frac{D}{2} = 2 \sin.^2 \frac{p + p}{2} - 2 \sin. p \,.\, \sin. p' \,.\, \cos.^2 \frac{P}{2}.$$

Let $A = \frac{p + p'}{2}$, and $\sin.^2 B = \sin. p \,.\, \sin. p' \,.\, \cos.^2 \frac{P}{2}$.

This last supposition is warranted, because $\sin. p$, $\sin. p'$, and $\cos.^2 \frac{P}{2}$ are each proper fractions, and therefore an angle can be found whose sine shall be equal to the product of these three quantities. Then—

$$\begin{aligned} \sin.^2 \frac{D}{2} &= \sin.^2 A - \sin.^2 B \\ &= \sin.^2 A \,(1 - \sin.^2 B) - \sin.^2 B \,(1 - \sin.^2 A) \\ &= \sin.^2 A \,.\, \cos.^2 B - \cos.^2 A \,.\, \sin.^2 B \\ &= \sin. (A + B) \,.\, \sin. (A - B). \end{aligned}$$

Hence we have—

$$\left.\begin{aligned} A &= \tfrac{1}{2} (p + p') \\ \sin. B &= \sqrt{\sin. p \,.\, \sin. p' \,.\, \cos.^2 \frac{P}{2}} \\ \sin. \frac{D}{2} &= \sqrt{\sin. (A + B) \,.\, \sin. (A - B)} \end{aligned}\right\} \quad . \quad . \quad . \quad \text{I.}$$

Secondly In the same triangle we have now the three sides given to find the angle $P X X'$, or the angle at the object farthest from the meridian. In all works on spherical trigonometry the following formula is proved :—

$$\begin{aligned} \cos. \frac{P X X'}{2} &= \sqrt{\frac{\sin. S \,.\, \sin. (S - p')}{\sin. p \,.\, \sin. D}} \\ &= \sqrt{\sin. S \,.\, \sin. (S - p') \,.\, \text{cosec.}\, p \,.\, \text{cosec.}\, D} \,. \quad \text{II.}, \end{aligned}$$

when S is half the sum of the three sides of the triangle $P X X'$.

Thirdly. In the triangle $Z X X'$ we have the three sides given, viz. the two zenith distances and D, to find the angle $Z X X'$ or the angle at the object farthest from the meridian. Using the same formula as in II., we have—

$$\begin{aligned} \cos. \frac{Z X X'}{2} &= \sqrt{\frac{\sin. S' \,.\, \sin. (S' - z')}{\sin. z \,.\, \sin. D}} \\ &= \sqrt{\sin. S' \,.\, \sin. (S' - z') \,.\, \text{cosec.}\, z \,.\, \text{cosec.}\, D} \,. \quad \text{III.}, \end{aligned}$$

when S' is half the sum of the three sides of the triangle $Z X X'$.

Fourthly. If the great circle $X X'$ which joins the two objects be produced, and pass between the zenith and the elevated pole, as in fig. 49, then the angle $P X Z$ is the sum of $P X X'$ and $Z X X'$; but when the great circle joining the two objects does not so pass, as in fig. 48, the difference between the angles $P X X'$ and $Z X X'$ will be the angle $P X Z$. In practice the objects can *always* be so chosen as to be quite free from ambiguity; and by drawing a figure as accurately as possible (which should always be done), the student can generally see for himself whether $P X Z = P X X' + Z X X'$ or $P X X' - Z X X'$. If an uncertainty still exists, it can only be removed by calculating on both suppositions and selecting the one nearest to the latitude by account.

Now, in the triangle $P X Z$ we have given the two sides $P X$ and $Z X$ and the included angle $P X Z$ to find the side $P Z$ or colatitude. This is solved precisely as in I., being the same operation with another triangle. The process may be thus expressed :—

$$\left.\begin{aligned} A' &= \tfrac{1}{2}(p+z) \\ \sin. B' &= \sqrt{\sin. p \,.\, \sin. z \,.\, \cos.^2 \frac{P X Z}{2}} \\ \sin. \frac{P Z}{2} &= \sqrt{\sin. (A'+B') \,.\, \sin. (A'-B')} \end{aligned}\right\} \quad . \quad . \quad \text{IV.}$$

Having obtained $\frac{P Z}{2}$, multiply it by two, and subtract the product from 90°. This completes the problem by giving the latitude.

The following rule is deduced from the above formulæ :—

RULE.—(*a*) Find the Greenwich time of both observations.

(*b*) Find the declinations, and thence the polar distances p and p'; take half the sum of the polar distances : call this A.

(*c*) Find the correction for the run during the interval.

(*d*) Correct the altitude at each observation, and find the zenith distances, z and z'.

(*e*) Obtain the polar angle by the rules already given and halve it, $\frac{P}{2}$.

(*f*) Then add together—

Twice log. cosine of half the polar angle . . $\cos.^2 \frac{P}{2}$,

log. sine of each polar distance . . sin. p and sin. p'; divide the sum by 2: this is log. sine of an arc, which call B.

Find the sum and difference of A and B, then half the sum of log. sin. $(A + B)$ and log. sin. $(A - B)$ will give the log. sine of half the distance between the object at the two observations, sin. $\frac{D}{2}$. The double of this we call Arc I.

(*g*) Add together:—

Polar distance of object at greater altitude . p'
„ „ less „ . . p
and Arc I.

Take half their sum (S), and subtract the polar distance of the object at greater altitude from this half sum ($S - p'$). Then add together—

log. cosec. p. d. of the object at less altitude . cosec. p,
log. cosec. Arc I. . . . . cosec. D,
log. sin. of half sum . . . . sin. S,
and log. sin. of remainder . . . . sin. $(S-p')$;

half this sum gives log. cosine of an arc, which call Arc II., cos. $\frac{P X X'}{2}$.

(*h*) Add together—

Zenith distance of object at greater altitude . . z',
„ „ less „ . . z,
and Arc. I.

Take half their sum S', and subtract the zenith distance of the object at greater altitude from this half sum ($S' - z'$). Then add together—

log. cosec. z. d. of the object at the less altitude . cosec. z,
log. cosec. Arc I. . . . . cosec. D,
log. sine of half the sum . . . . sin. S',
log. sin. of the remainder . . . . sin.$(S'-z')$;

half this sum gives the log. cosine of an arc, which call Arc III., cos. $\frac{Z X X'}{2}$.

(*i*) If the great circle which joins the positions of the object or objects at the two observations pass

between the zenith and the elevated pole, then add together Arc II. and Arc III.; but if the great circle do not so pass, subtract Arc III. from Arc II.; call the sum or difference Q.

(*k*) Take half the sum of the polar distance and zenith distance at the less altitude $\frac{p+z}{2}$; call this A'.

(*l*) Then add together—

Twice log. sine of Q	$\cos.^2 \frac{PXZ}{2}$,
log. sine p. d. at the less altitude	$\sin. p$,
and log. sine z. d. at „ „	$\sin. z$;

divide the sum by 2; this is the log. sine of an arc, which call B'. Find the sum and difference between A' and B', then half the sum of log. sin. $(A' + B')$ and log. sin. $(A' - B')$ will give the log. sine of half the colatitude $\sin. \frac{PZ}{2}$. The double of this subtracted from 90° gives the latitude.

(*m*) Having found the latitude, the longitude is deduced, as already shown, from the least altitude.

Ex. 457. 1887, February 28th, in latitude by account 47° 20′ N., longitude 133° 10′ W., the following observations were made:—

App. time ship nearly.	*Chronometer.*	*Observed altitudes.*
9h. 53m. a.m.	6h. 32m. 3s.	$\underline{\odot}$ 27° 55′ 0″
0 56 p.m.	9 35 15	$\underline{\odot}$ 33 32 0

The sun's magnetic bearing at the first observation was S.E. by E. $\frac{1}{4}$ E., and the run of the ship in the interval was magnetic S. by E. $\frac{3}{4}$ E., at the rate of 6 miles per hour. The chronometer was slow of G. M. T. on 4th January, 27m. 10s., and gaining daily ·6s. The index error of the sextant was $-$ 2′ 19″, and the height of the eye 23 feet. Required the latitude and longitude of the ship at the time of taking the less altitude.

For Greenwich apparent times nearly.

	First observation.	*Second observation.*
February	27d. 21h. 53m. 0s.	Feb. 28d. 0h. 56m. 0s.
Long. in time	+ 8 52 40	8 52 40
Greenwich app. time	28 6 45 40	Feb. 28 9 48 40

For correct Greenwich mean time.

First observation.				*Second observation.*			
Time by chronometer	6h.	32m.	3s.		9h.	35m.	15s.
Slow	+	27	10		+	27	10
	6	59	13		10	2	25
Gain 55·29 × ·6		−	33·17	Gain 55·42 × ·6		−	33·25
G. M. T., Feb. 28	6	58	39·83	Feb. 28	10	1	51·75

For P. D., first observation.		*Variation of declination.*	
Sun's declination	7° 58′ 6·0 S.	For 1 hour	− 56·69″
Correction	− 6 35·7	No. hours	6·98
True declination	7 51 30·3 S.		60)395·6962
∴ N. P. D.	97 51 30·3 . . p	Cor.	− 6 35·7

For P. D., second observation.		*Variation of declination.*	
Sun's declination	7° 58′ 6·0 S.	For 1 hour	− 56·69″
Correction	− 9 28·7	No. hours	10·031
True declination	7 48 37·3 S.		60)568·65739
∴ N. P. D.	97 48 37·3 . . p'	Cor.	− 9 28·7

For eq. time, first observation.		*Variation of E. T.*	
Equation of time	− 12m. 45·05s.	In 1 hour	− ·470s.
Correction	− 3·28	No. hours	6·98
True equation of time	− 12 41·77	Correction	3·28080

For eq. time, second observation.		*Variation of E. T.*	
Equation of time	− 12m. 45·05s.	In 1 hour	− ·470s.
Correction	− 4·71	No. hours	10·031
True equation of time	− 12 40·34	Correction	4·714570

Correction for run.

Sun's mag. bearing	S.E. by E.¼E.	Included angle 3½ pts. cos.	9·888185
Ship's course	S. by E.¾E.	Run 3·053 × 6 = 18·322′ log.	1·262925
Included angle	3½ pts.	Cor. for run 14·16′ log.	1·151110

For Z. D., first observation.

Observed alt. L. L.	27° 55′ 0″
Index error	— 2 19
	27 52 41
Dip	— 4 43
	27 47 58
Semidiameter	+ 16 10
	28 4 8
Refrac. and parallax	— 1 38
True alt.	28 2 30
Zenith distance	61 57 30 .. z

For Z. D., second observation.

Observed alt. L. L.	33° 32′ 0″
Index error	— 2 19
	33 29 41
Dip	— 4 43
	33 24 58
Semidiameter	+ 16 10
	33 41 8
Refrac. and parallax	— 1 18
	33 39 50
Correction for run	— 14 10
True alt.	33 25 40
Zenith distance	56 34 20 .. z'

For half polar angle.

G. M. T., first obser.	6h. 58m. 39·83s.
„ second „	10 1 51·75
	3 3 11·92
Correction for E. T.	+ 1·43
Polar angle	3 3 13·35
	= 45° 48′ 20·25″
$\therefore \frac{P}{2}$	= 22 54 10

Var. E. T. 1 hr.	·470s.
Elapsed time	3·0503
Correction	1·434910

For Arc I.

$\frac{P}{2}$	22° 54′ 10″	cos.	9·964338
			2
		cos.²	19·928676
p	97 51 30	sin.	9·995902
p'	97 48 37	sin.	9·995952
A	97 50 4		2)19·920530
B	65 51 45	sin.	9·960265
Sum	163 41 49	sin.	9·448271
Diff.	31 58 19	sin.	9·723869
			2)19·172140
$\frac{D}{2}$	22° 40′ 38″	sin.	9·586070
	2		
D	45 21 16 = Arc I.		

For Arc III.

z'	56° 34′ 20″		
z	61 57 30	cosec.	·054233
D	45 21 16	cosec.	·147845
	2)163 53 6		
S'	81 56 33	sin.	9·995691
$S' - z'$	25 22 13	sin.	9·631917
			2)19·829686
Arc. III.	34° 43′ 11″	cos.	9·914843
Arc. II.	46 37 8		
Q	11 53 57		

For Arc II.

p'	97° 48′ 37·3″		
p	97 51 30·3	cosec.	·004098
D	45 21 16	cosec.	·147845
	2)241 1 24		
S	120 30 42	sin.	9·935268
$S-p'$	22 42 5	sin.	9·586507
			2)19·673718
Arc II.	46° 37′ 8″	cos.	9·836859

For colatitude.

Q	11° 53′ 57″	cos.	9·990566
			2
			19·981132
p	97 51 30	sin.	9·995902
z	61 57 30	sin.	9·945767
A'	79 54 30		2)19·922801
B'	66 11 58·6	sin.	9·961400
Sum	146 6 28·6	sin.	9·746346
Diff.	13 42 31·4	sin.	9·374723
			2)19·121069
$\frac{Colat.}{2}$	21° 19′ 1″	sin.	9·560534
Colat.	42 38 2		
Lat.	47 21 58 N.		

The student should notice that Arc I. and half the colatitude are found by a similar process, as are also Arcs II. and III. If a blank form be first written down, the work will be much facilitated, because many logarithms are taken from the same opening of the tables. Having now found the correct latitude, the longitude is found in the usual manner.

For longitude at first observation.

a	28° 2′ 30″		
l	47 21 58	sec.	·169202
p	97 51 30	cosec.	·004098
	2)173 15 58		
S	86 37 59	cos.	8·768864
$s-a$	58 35 29	sin.	9·931189
			2)18·873353
$\frac{h}{2}$	15 51 43·6	sin.	9·436676
	2		
h	31 43 27·2		
	4		
	60)126 53 48·8		

Eastern hour angle	2h.	6m.	53·81s.	
	24			
App. time ship, Feb.	27d.	21h.	53m.	6·19s.
Equa. time		+	12	41·77
M. T. ship, Feb.	27	22	5	47·96
M.T. Greenwich, Feb.	28	6	58	39·83
Long. in time		8	52	51·87
		60		
		4)532		51·87
Longitude		133°	12′	58″ W.

This is the longitude at the place where the first observation was made, because we have used the altitude as taken at that station.

EXERCISE XIX.

Ex. 458. 1887. On December 24th, in latitude by account 47° 30′ N., longitude 8° 30′ W., the following observations were made:—

Approx. time ship.	*Time by chronometer.*	*Observed altitudes.*
11h. 15m. a.m.	0h. 16m. 6s.	☉̄ 18° 21′ 20″
2 30 p.m.	3 31 4	☉̄ 11 43 45

The sun's compass-bearing at the time of taking the first observation was S. by W., and the run of the ship in the interval between the observations was 30 miles on a S.W. by W. ¾ W. course. The chronometer was fast 27m. 19s. of G. M. T. on 17th of November, and losing daily ·7s. The index correction of the sextant was − 1′ 17″, and the height of the eye above the sea was 20 feet. Required the latitude and longitude of the place at the time of taking the second observation.

Ex. 459. 1887, September 1st, at ship by reckoning in latitude 49° N., longitude 180° E., the following observations were made:—

Approx. time ship.	*Time by chronometer.*	*Observed altitudes.*
0h. 33m. p.m.	0h. 42m. 57s.	☉̲ 48° 51′ 0″
4 41 p.m.	4 50 20	☉̄ 19 30 20

The sun's bearing at the first observation was S. by W., and the run of the ship in the interval was E.N.E., both by compass, at the rate of 6½ knots per hour. The chronometer was fast 5m. 13s. on 12th May for G. M. T., and gaining daily 2s. The height of the eye above the sea was 17 feet, and the index error of the sextant was + 1′ 10″. Find the ship's position at the second observation.

Ex. 460. 1887, March 20th, in latitude by account 26° 20′ S., longitude 115° 25′ W., the following observations were made:—

Approx. ship app. time.	*Time by chro.*	*Observed altitudes.*
11h. 18m. a.m.	6h. 51m. 22s.	☉̲ 62° 7′ 20″
3 40 p.m.	11 5 10	☉̲ 30 31 40

The index correction of the sextant was + 3′ 24″, and height of eye above the sea 22 feet. The chronometer was slow of G. M .T. on 23rd February, 23m. 39s., and losing daily 1·4s. The sun's bearing at the first observation by compass was

S.E. by S., and the ship's run by compass during the interval was S.W. by W., $7\frac{3}{4}$ knots per hour. Required the ship's position at the second observation.

+ Ex. 461. 1887, June 27th. With a sextant whose index error was — 7′ 18″, at a height above the sea of 15 feet, the following observations were made in latitude by account 15° 10′ S., and longitude 78° 20′ E. :—

App. time ship, approx.	*Time by chro.*	*Observed altitudes.*
8h. 30m. a.m.	3h. 17m. 15s.	☉̲ 26° 50′ 10″
11 40 a.m.	6 26 1	☉̅ 51 6 15

The bearing of the sun at the first observation was S. 60° E., and course during the interval N.E., both by compass, at the rate of $7\frac{1}{2}$ knots per hour. The chronometer was slow on 28th of April, 5m. 16s. of G. M. T., and gaining daily ·4s. Required the position of the ship at the first observation.

Ex. 462. 1887, May 30th, at ship, the position by account was 7° 30′ N. latitude, and 168° 40′ W. longitude, and the height of the eye above the sea was 24 feet. The first altitude was taken with a sextant whose index error was —1′ 1″, and the second with one whose index error was + 3′ 20″.

App. time ship nearly.	*Time by chro.*	*Observed altitudes.*
0h. 24m. p.m.	11h. 32m. 52s.	☉̅ 74° 18′ 50″
4 20 p.m.	3 28 20	☉̲ 26 21 50

The chronometer on 3rd April was slow of G. M. T. 4m. 32s., and gaining daily ·7s. The sun's bearing at the latter observation was N. 73° W., and the ship's course in the interval between the observations was N.N.E., both by compass, at the rate of $5\frac{3}{4}$ knots per hour. Find the ship's position at the second observation.

Ex. 463. 1887, November 29th, the following observations were made in latitude by account 6° 40′ S., longitude 179° 50′ E. :—

App. time ship nearly.	*Time by chro.*	*Observed altitudes.*
7h. 40m. a.m.	7h. 59m. 42s.	☉̲ 25° 44′ 45″
11 16 a.m.	11 34 30	☉̅ 71 32 45

The chronometer was on 19th September fast of G. M. T. 32m. 29s., and losing daily ·4s. The index error of the sextant was + 1′ 44″, and height of the eye above the sea 21 feet. The compass bearing of the sun at the first observation was S. 60° E., and the compass course in the interval between

the observations was S. 60° E., at the rate of $4\frac{3}{4}$ knots per hour. Required the position of the ship at the first observation.

Ex. 464. 1887, July 31st, at ship in latitude by account 30° 26′ N., longitude 132° 30′ E., the following observations were made :—

App. time at ship nearly.	*Time by chro.*	*Observed altitudes.*
9h. 17m. p.m.	11h. 30m. 11s.	Vega 79° 1′ 40″
2 29 a.m.	4 41 58	Vega 33 38 50

The star at the second observation bore W.N.W., and the ship's course in the interval was due South, both by compass, at the rate of $6\frac{1}{2}$ knots per hour. The chronometer on 16th June was slow 1h. 2m. 53s. of G. M. T., and losing daily ·4s. The index correction was − 2′ 29″, and height of the eye 23 feet. Find the latitude and longitude at the second observation.

Ex. 465. 1887, February 14th p.m., the position of the ship by account was 50° 20′ N. latitude and 39° 20′ W. longitude. The chronometer showed 0h. 54m. 22s., and was slow on 13th January, 40m. 51s., and losing 3·5s. daily. The altitude of Procyon W. of the meridian was observed to be 43° 12′ 20″, index error − 2′ 11″. At the same instant by another observer, the altitude of Arcturus E. of meridian was 19° 36′, index error + 1′ 29″, height of the eye 21 feet. Required the ship's position,

Ex. 466. April 8th a.m., in latitude by account 45° 40′ S., longitude 68° 40′ E., the observed altitude of Spica W. of the meridian was 13° 7′ 0″ at about 5h. 45m. a.m. apparent time at ship; and about 5h. 46m. 40s. the observed altitude of Altair E. of the meridian was 34° 23′ 0″. The chronometer at the latter observation showed 12h. 4m. 7s., which on the 26th February had been found 1h. 9m. 14s. slow of G. M. T., and gaining 3·2s. daily; index error + 1′ 14″, height of the eye 25 feet. Required the position of the ship.

IVORY'S METHOD OF DOUBLE ALTITUDES.

This method, strictly speaking, can be applied only to bodies which do not change their declinations between the observations, such as the fixed stars; but by introducing corrections to the latitude and longitude derived from the altitudes of other bodies, it will be found to produce results as exact as can be obtained at sea by other methods. It was proposed by the late James

Ivory, Professor of Mathematics at the Royal Military College at Sandhurst.

Proof.—Let $N\ E\ S\ W$ be the projection of the celestial concave on the plane of the horizon, $N\ S$ the meridian, P the

Fig 50.

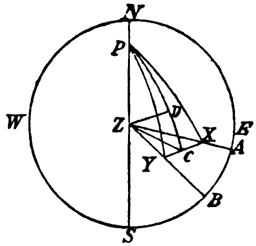
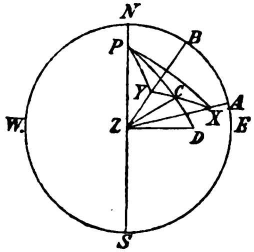


Fig 51



elevated pole, Z the zenith, X the position of the object at the first observation, Y at the second. Join $P\ X$, $P\ Y$; then $P\ X$, $P\ Y$ are the polar distances, and in this method these are considered equal to that at the middle time between the observations; $A\ X$, $B\ Y$ the altitudes, and therefore $Z\ X$, $Z\ Y$ are the zenith distances. Join $X\ Y$ by an arc of a great circle, draw $P\ C$ bisecting the angle $X\ P\ Y$, and therefore also bisecting $X\ Y$ at right angles; and draw $Z\ D$ perpendicular to $P\ C$, or to $P\ C$ produced join $Z\ C$.

Then $X P Y$ is the measure of the elapsed time P, and hence $X\ P\ C$, $Y\ P\ C$ measure half the elapsed time $\frac{P}{2}$.

We shall denote $X\ C = Y\ C$ by Arc I.; $C\ P$ by Arc II.; $Z D$ by Arc III; $C\ D$ by Arc IV.; and $D\ P$ by Arc V. $Z\ P$ is the colatitude.

In the right-angled spherical triangle $P\ C\ X$

$$\sin.\ X P\ .\ \sin.\ X P\ C = \sin.\ X C,$$

$$\sin.\ p\ .\ \sin.\ \frac{P}{2} = \sin.\ \text{Arc I.};$$

$$\therefore\ \sin.\ \text{Arc I.} = \sin.\ \frac{P}{2}\ .\ \cos.\ \text{dec.},$$

$$\text{or cosec. Arc I.} = \text{cosec.}\ \frac{P}{2}\ .\ \text{sec. dec.} \quad . \quad . \quad . \quad (1).$$

In the same triangle

$$\cos. PX = \cos. PC . \cos. XC,$$
$$\cos. PC = \cos. PX . \sec. XC,$$
$$\text{or } \cos. \text{Arc II.} = \sin. \text{dec.} . \sec. \text{Arc I.} \quad . \quad . \quad . \quad (2).$$

Because PCX and PCY are right angles, therefore $\cos. ZCY = \sin. ZCD$, and $\cos. ZCX = -\sin. ZCD$; and hence in the spherical triangles, ZCX, ZCY

$\cos. ZX$. or $\sin. AX$

$$= \cos. ZC . \cos. XC - \sin. ZC . \sin. XC . \sin. ZCD \quad . \quad (\alpha)$$

$\cos. ZY$ or $\sin. BY$

$$= \cos. ZC . \cos. YC + \sin. ZC . \sin. YC . \sin. ZCD \quad . \quad (\beta),$$

and $XC = YC$; therefore, by subtraction,

$$\sin. BY - \sin. AX = 2 \sin. ZC . \sin. YC . \sin. ZCD;$$
$$\text{but } \sin. ZC . \sin. ZCD = \sin. ZD.$$

Hence, $\sin. BY - \sin. AX = 2 \sin. ZD . \sin. YC$,

$$\text{or } \sin. ZD = \frac{\sin. BY - \sin. AX}{2 \sin. YC}$$
$$= \frac{2 \cos. \frac{BY + AX}{2} . \sin. \frac{BY - AX}{2}}{2 \sin. YC}.$$

Let $\frac{s}{2}$ = half the sum of altitudes $= \frac{BY + AX}{2}$,

and $\frac{d}{2}$ = half the diff. of altitudes $= \frac{BY - AX}{2}$.

$$\text{Then } \sin. \text{Arc III.} = \text{cosec. Arc I.} \cos. \frac{s}{2} . \sin. \frac{d}{2} \quad . \quad . \quad (3)$$

Next, by adding (α) and (β),—

$$\sin. BY + \sin. AX = 2 \cos. ZC . \cos. YC,$$
$$\text{but } \cos. ZC = \cos. ZD . \cos. CD;$$
$$\therefore \sin. BY + \sin. AX = 2 \cos. CD . \cos. ZD . \cos. YC;$$
$$\cos. CD = \frac{\sin. BY + \sin. AX}{2 \cos. ZD . \cos. YC}$$
$$= \frac{2 \sin. \frac{BY + AX}{2} . \cos. \frac{BY - AX}{2}}{2 \cos ZD . \cos. YC}.$$

$$\text{Hence } \cos. \text{Arc IV.} = \sec. \text{Arc I.} \sin. \frac{s}{2} . \cos. \frac{d}{2} . \sec. \text{Arc III.} \ (4).$$

In the case represented by fig. 50, $PD = PC - DC$; in that represented by fig. 51, $PD = PC + DC$; and $\therefore$ Arc V. $=$ Arc II. $\mp$ Arc IV. . . . . (5).

In the spherical triangle $P\ Z\ D$ right-angled at D

$$\cos. P Z = \cos. P D . \cos. Z D,$$

$$\text{sin. lat.} = \text{cos. Arc V.} . \text{cos. Arc III.},$$

$$\text{or cosec. lat.} = \text{sec. Arc V.} . \text{sec. Arc. III.} \quad . \quad (6)$$

The formulæ marked 1 to 6 are those used in calculating the latitude by Ivory's method.

The late Edward Riddle, Head-master of the Royal Hospital Nautical School, Greenwich, extended the problem so as to obtain the apparent time at ship at the middle time between the two observations; and hence the longitude could be ascertained if the times be taken by a chronometer whose error and rate are known. His formula is obtained as follows:—

The arcs $P\ Z$ or colatitude, and $D\ Z$ or Arc III., are known in the right-angled triangle $P\ Z\ D$, and

$$\sin. D Z = \sin. Z P . \sin. Z P D;$$

$$\therefore \sin. Z P D = \sin. D Z . \text{cosec.}\ Z P.$$

Hence sine hour angle at mid time = sin. Arc III. sec. lat. (7)

From the above formulæ the following rule is deduced:—

Rule (*a*) Find the Greenwich time corresponding to the middle time between the observations.

(*b*) Find the polar angle P, the same as for the direct method, divide it by two, and reduce it to degrees, &c.

(*c*) Find the declination for Greenwich time found in (*a*).

(*d*) Calculate the correction for run to be applied as directed in pages 230-1.

(*e*) Correct the altitudes, take half their sum $\frac{s}{2}$, and half their difference $\frac{d}{2}$.

(*f*) *For Arc I.*—To log. cosec. half elapsed time, add log. sec. declination; the sum is log. cosec. Arc I.

$$\text{Cosec. Arc I.} = \text{cosec.}\ \frac{P}{2} . \text{sec. dec.}$$

(*g*) *For Arc II.*—To log. sin. dec. add log. sec. Arc I.; the sum is log. cos. Arc II.

$$\text{Cos. Arc. II.} = \text{sin. dec. sec. Arc. I.}$$

When latitude and declination are of different names, the supplement of this angle is Arc II.

(*h*) *For Arc III.*—To log cosec. Arc I. add log. cos. half the sum of altitudes, and log. sin. of half the difference of altitudes; this sum is log. sin. Arc III.

$$\text{Sin. Arc III.} = \text{cosec. Arc I.} . \cos. \frac{s}{2} . \sin. \frac{d}{2}.$$

(*i*) *For Arc IV.*—To log. sec. Arc I. add log. sin. half the sum of the altitudes, log. cos. half the difference of altitudes, and log. sec. Arc III.; the sum is log. sin. Arc IV.

$$\text{sin. Arc IV.} = \text{sec. Arc I.} \cdot \text{sin.} \frac{s}{2} \cdot \text{cos.} \frac{d}{2} \cdot \text{sec. Arc III.}$$

(*k*) *For Arc V.*—There is some uncertainty about this arc, which has already been alluded to when treating of the direct method. This will be partially removed by attending to the following precepts:—

(A) When the latitude by account and the declination are of the same name.

(*a*) *If the observations be taken on opposite sides of the meridian*, and the declination be less than the latitude by account, take the difference between Arc II. and Arc IV. for Arc V.; if the declination be greater than the latitude by account, take the sum of Arc II. and Arc IV. for Arc V.

(β) *If the observations be taken on the same side of the meridian*, and the declination be less than the latitude by account, take the difference between Arc II. and Arc IV. for Arc V.; if the declination be greater than the latitude by account, or if it be uncertain which is the greater, then find Arc V. by both adding and subtracting Arcs II. and IV. Calculate the latitudes by (*l*), and select the result nearest to the latitude by account. These ambiguities can only occur, in taking the sun, between the tropics: a *star* can always be so selected as to free the problem from all uncertainty.

(B) If the latitude and declination be of contrary names, Arc V. is always equal to Arc II. – Arc IV.

(*l*) *For latitude.*—To log. sec. Arc V. add log. sec. Arc III.; this gives log. cosec. latitude.

Cosec. lat. = sec. Arc V. sec. Arc III.

(*m*) *If longitude be required.*—To log. sin. Arc III. add log. sec. lat., the sum is log. sin. hour angle at the middle time nearly.

Sin. hour angle at mid time = sin. Arc. III. sec. lat.

It should be borne in mind that the smaller *P*, or the elapsed time, the greater is the care required

in reducing the data, and in taking out the logarithms.

Ex. 467. 1887, February 28th, in latitude by account 47° 20′ N., longitude 133° 10′ W., the following observations were made:—

App. time at ship nearly.	*Chronometer.*	*Observed altitudes.*
9h. 53m. a.m.	6h. 32m. 3s.	☉ 27° 55′ 0″
0 56 p.m.	9 35 15	☉ 33 32 0

The sun's magnetic bearing at the first observation was S.E. by E. $\frac{1}{4}$ E., and the run of the ship in the interval was S. by E. $\frac{3}{4}$ E., magnetic, at the rate of six miles per hour. The chronometer was slow of G. M. T. 27m. 10s. on January 4th, and gaining daily ·6s. The index error of the sextant was − 2′ 19″, and the height of the eye 23 feet. Required the latitude and longitude of the ship at the time of taking the less altitude.

For Greenwich apparent time nearly.

First observation.

	d.	h.	m.	s.
February	27d.	21h.	53m.	0s.
Long. in time	+	8	52	40
Greenwich app. time	28	6	45	40

Second observation.

	d.	h.	m.	s.
February	28d.	0h.	56m.	0s.
	+	8	52	40
February	28	9	48	40

For correct Greenwich mean time.

First observation.

		h.	m.	s.
Time by chronometer		6h.	32m.	3s.
Slow		+	27	10
		6	59	13
Gain 55·29 × ·6			−	33·17
G. M. T.	Feb. 28	6	58	39·83

Second observation.

		h.	m.	s.
		9h.	35m.	15s.
		+	27	10
		10	2	35
Gain 55·42 × ·6			−	33·25
	Feb. 28	10	1	51·75

For G. M. T. at mid observation.

	h.	m.	s.
G. M. T. first obser.	6h.	58m.	39·83s.
G. M. T. second „	10	1	51·75
	2)17	0	31·58
G. M. T. Feb. 28	8	30	16

For half polar angle.

	h.	m.	s.
	6h.	58m.	39·83s.
	10	1	51·75
Interval in M. T.	3	3	11·92
Cor. for eq. time		+	1·43
Int. in app. time	3	3	13·35
Polar angle	45°	48′	20·25″
$\therefore \frac{P}{2}$ =	22	54	10

For sun's declination at mid time.

February 28	7° 58′ 6·0″ S.
Correction	− 8 1·9
True declination	7 50 4·1 S.

Variation of declination.

For 1 hour	− 56·69″
No. hours	8·5
Correction	− 481·865

For eq. time at mid time.

February 28	+ 12m. 45·05s.
Correction	− 4·00
True eq. time	+ 12 41·05

Variation of eq. time.

For 1 hour	·470s.
No. hours	8·5
Correction	3·9950

Correction for run.

Sun's mag. bearing	S. 5¼ E.
Ship's course	S. 1¾ E.
Included angle	3½ pts.

Included angle 3½ pts.	cos.	9·888185
Run 3·053 × 6 = 18·322′	log.	1·262925
Correct. for run 14·16′	log.	1·151110
= 14′ 10″		

Because the run has been towards the sun, the correction for the second observation is − 14′ 10″.

For alt. first observation.

Observed alt. L. L.	27° 55′ 0″
Index error	− 2 19
	27 52 41
Dip	− 4 43
	27 47 58
Semidiameter	+ 16 10
	28 4 8
Refrac. and parallax	− 1 38
First true alt.	28 2 30
Second „	33 25 40
	2)61 28 10
Half sum alts.	30 44 5 $\left(\frac{s}{2}\right)$

For alt. second observation.

Observed alt. L. L.	33° 32′ 0″
Index error	− 2 19
	33 29 41
Dip	− 4 43
	33 24 58
Semidiameter	+ 16 10
	33 41 8
Refrac. and parallax	− 1 18
	33 39 50
Correction for run	− 14 10
Second reduced alt.	33 25 40
First „	28 2 30
	2)5 23 10
Half diff. alts.	2 41 35 $\left(\frac{d}{2}\right)$

To find the latitude.

$\frac{P}{2}$	22°	54′	10″	cosec. ·409862					
Dec.	7	50	4	sec. ·004073				sin. 9·134531	
Arc I.	22	40	38	cosec. ·413935	sec. ·034943			sec. ·034943	
$\frac{s}{2}$	30	44	5	cos. 9·934268	sin. 9·708476				
$\frac{d}{2}$	2	41	35	sin. 8·671961	cos. 9·999520				
Arc III.	6	0	46	sin. 9·020164	sec. ·002396				
Arc IV.	56	11	5	. .	cos. 9·745335				
Arc II.	98	29	44	. . . .	81° 30′ 16″			cos. 9·169470	
Arc V.	42	17	53	sec. ·130971					
Arc III.	6	0	46	sec. ·002396					
Latitude	47	21	24	cosec. ·133367					

For longitude.

Arc III.	6°	0′	46″	sin. 9·020155
Latitude	47	21	24	sec. ·169135
Hour angle at mid time	8	53	43	sin. 9·189290
			4	
	35	34	52	
Eastern hour angle	0h.	35m.	34·87s.	
	24			
App. time ship, Feb. 27	23	24	25·13	
Equation time	+	12	41·05	
Mean time ship, Feb 27	23	37	6·18	
M. T. Greenwich, Feb. 28	8	30	15·79	
Longitude	8	53	9·61	
	60			
	4)533	9·61		
Longitude	133	17	24 W.	

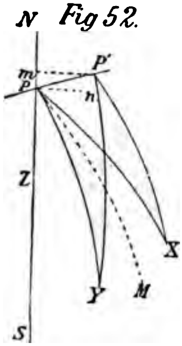
By comparing the latitude and longitude here found with the

results given by the *direct method*, it will be seen there is a discrepancy. This arises from considering the sun's declination to be unchanging and equal to that at the middle time of observation. We shall now proceed to show how corrections are made for this assumption.

MODIFICATIONS NECESSARY.—If the object be either the sun, moon, or planets, it changes its declination during the interval, and, for strict accuracy, small corrections are necessary for the latitude and longitude found by Ivory's method. In practice these corrections may be disregarded, because they are so small, and also because one of the altitudes must be simply an approximation arising from the reduction due to the run. The corrections referred to above are found as follows.

Let NS be a meridian, N and S the north and south points of the horizon; Z the zenith, X and Y the sun at the times of observation, P the assumed pole at these times, so that PX and PY are equal. Let P' be the true position of the pole given by the true polar distances at the times of observation. Join $P'Y$ and $P'X$; then, if the polar distance be increasing, $P'Y$ is greater than PY; make $Yn = XP$ and join nP, then nP' is the change in the polar distance, and nP may be considered at right angles to YP and to YP'. Join PP', then PP' will be at right angles to PM, the hour circle at the mid time of observations; and being so small (never more than a few miles), may be considered a straight line. From P' draw $P'm$ at right angles to the meridian; then, because NP is the latitude by calculation, Pm is the correction for the latitude.

Fig 52.



Now $mPP' + P'PM + MPS = 2$ right angles,

and $P'PM$ is a right angle;

$\therefore\ mPP'$ is complementary to MPS . . (*a*)

Again $MPP' = YPn$, each being a right angle;

take away the common part MPn;

the remainder $P'Pn =$ remainder MPY . (*b*),

and $Pm = PP' \,.\, \text{cos.}\, P'Pm$

$= P'n \,.\, \text{cosec.}\, P'Pn \,.\, \text{cos.}\, P'Pm$

from (*a*) and (*b*) $= P'n \,.\, \text{cosec.}\, MPY \,.\, \text{sin.}\, MPS.$

Now, as the declination used in the solution of the problem

is that for the mid time, the change in declination $P'n$ must be that due to half the elapsed time.

Hence, cor. for lat. = change in dec. for half the elapsed time, × cosec. half elapsed time × sin. hr. angle at mid time.

In the diagram above we see

(*a*) The sun is nearer the meridian at the second than at the first observation, and its altitude is therefore greater than at the first.

(*b*) The polar distance $P'Y$ at the second observation is greater than $P'X$, the polar distance at the first, hence the days are decreasing in length; and

(*c*) Nm, the corrected latitude, is less than NP, found by considering the polar distances equal; the correction Pm is therefore subtractive from PN.

From these considerations we see that if the second altitude be greater than the first and the days be decreasing in length, the correction Pm must be subtracted from the computed latitude. If either of these conditions be changed the correction Pm must be added, but if both be changed the correction is still subtractive.

Rule (1) Multiply the hourly difference of the declination by half the elapsed time in hours; this gives *c*.

(2) Add together log. *c*, log. cosec. half the elapsed time, and log. sin. hour angle at the mid time; the sum is log. of correction of latitude in seconds.

(3) This correction must be *subtracted* when the second altitude is greater than the first and the days decreasing in length, or when the second altitude is less than the first and the days increasing in length; but must be *added* when the second altitude is greater than the first and the days increasing in length, or when the second altitude is less than the first and the days decreasing in length.

Ex. 468. Apply the foregoing rule to correct the latitude of the question worked out by Ivory's method.

To find the change in dec. in half elapsed time.

Variation of dec. for 1 hour	56·69″	
Half elapsed time	1·53	
Change in dec.	86·7357	. (*c*)

Cor. for lat. $=$ *c* $\times$ *cosec. half elap. time* $\times$ *sin. hr. angle at mid time.*

c	86·7357″	log. 1·938198
Half elap. time	22° 54′ 10″	cosec. ·409862
Hr. ∠ mid time	8 53 43	sin. 9·189291
Correction	+ 34·46″	log. 1·537351

Approx. lat. by Ivory's method	47° 21′ 24″ N.
Correction for latitude	+ 34·5
True latitude	47 21 58·5 N.

Thus agreeing exactly with the latitude found by the *direct method.*

The longitude also requires a correction, and for two reasons :—

First, because we have supposed the polar distance to remain constant in the interval between the observations and have used that at the middle time ; and,

Secondly, because in finding the hour angle the approximate latitude has been employed instead of the true one.

The correction for longitude from these two causes may be thus found. Beginning with the fundamental formula, we must remember that the hour angle will vary as the latitude and polar distance do ; then—

$$\cos z = \cos. l' \,.\, \cos. p + \sin. l' .\, \sin. p \,.\, \cos. h.$$
$$= \sin. l \,.\, \cos. p + \cos. l \,.\, \sin. p \,.\, \cos. h.$$

Differentiating—

$$0 = \cos. l \,.\, \cos. p \,.\, dl - \sin. l \,.\, \sin. p \,.\, dp - \sin. l \,.\, \sin. p \,.\, \cos. h \,.\, dl$$
$$+ \cos. l \,.\, \cos. p \,.\, \cos. h \,.\, dp - \cos. l \,.\, \sin. p \,.\, \sin. h \,.\, dh.$$
$$\cos. l \,.\, \sin. p \,.\, \sin. h \,.\, dh = dl\,(\cos. l \,.\, \cos. p - \sin. l \,.\, \sin. p \,.\, \cos. h)$$
$$- dp\,(\sin. l \,.\, \sin. p - \cos. l \,.\, \cos. p \,.\, \cos. h)\,;$$
$$\therefore dh = dl\,(\cot. p \,.\, \text{cosec. } h - \tan. l \,.\, \cot. h)$$
$$- dp\,(\tan. l \,.\, \text{cosec. } h - \cot. p \,.\, \cot. h),$$

where dh, the correction for the hour angle, is that due to the errors in latitude (dl) and polar distance (dp).

In using this formula it is necessary to attend to the signs of its different terms. Now, as the latitude increases the colatitude will decrease, and the hour angle will become greater, and *vice versâ:* hence, if the latitude be increasing the correction for this error is +, and if the latitude be decreasing the

correction is —. Again, as the polar distance becomes greater the hour angle becomes less, and *vice versâ.*

We will now apply the formula to the example already worked, in which the values are—

$dp = 86{\cdot}7357''$	$p = 97^\circ\ 50'\ 4''$	$h = 8^\circ\ 53'\ 43''$
$dl = 34{\cdot}46$	$l = 47\ \ 21\ \ 24$	

Correction of hour angle for error in latitude.

dl	34·46″	log.	1·537351	dl	34·46″	log.	1·537351
p	97° 50′ 4″	cot.	9·138604	l	47° 21′ 24″	tan.	·035767
h	8 53 43	cosec.	·810709	h	8 53 43	cot.	·805454
A	− 30·67″	log.	1·486664	B	− 239·1″	log.	2·378572

In the above result it will be seen we have marked A with a — sign because the polar distance is greater than 90°, and its cot. is therefore —

Hence correction for latitude $= 30{\cdot}67'' + 239{\cdot}1''$

$= +\ 269{\cdot}77$

+ because the latitude is increasing.

Correction for error in the polar distance.

dp	86·7357″	log.	1·938198	dp	86·7357	log.	1·938198
l	47° 21′ 24″	tan.	·035767	p	97° 50′ 4″	cot.	9·138604
h	8 53 43	cosec.	·810709	h	8 53 43	cot.	·805454
A'	− 609·1	log.	2·784674	B'	+ 76·25	log.	1·882256

In the second term of this correction, because there is a minus sign before it, and the cot. of the polar distance is also minus, we have marked it + ; and because the polar distance is decreasing the whole correction is +.

Hence correction for polar distance $= +\ (-\ 609{\cdot}1 + 76{\cdot}25)$

$= -\ 532{\cdot}85$

and therefore the correction of the hour angle for both errors

$$= 269{\cdot}77'' - 532{\cdot}85''$$
$$= -\ 263{\cdot}08$$
$$= -\ 4'\ 23''$$

If this amount be subtracted from the hour angle, it will also give the longitude 4′ 23″ less. Hence—

Approximate longitude	133° 17′ 24″ W.
Correction for longitude	— 4 23
True longitude	133 13 1 W.

which agrees with that obtained from the direct method within 3″.

The calculation of these corrections is so long, and the application requires so much care, that it will be found to be far easier to compute the longitude precisely as is done in the direct method, using the correct altitude, latitude, and polar distance for the observation named.

The student should compare the results obtained by Ivory's method with those already obtained by the direct method, and also with those following by Sumner's method.

EXERCISE XX.

Ex. 469. 1887. On December 24th in latitude by account 47° 30′ N., longitude 8° 30′ W., the following observations were made :—

App. time ship nearly.	*Time by chro.*	*Observed altitudes.*
11h. 15m. a.m.	0h. 16m. 6s.	☉̄ 18° 21′ 20″
2 30 p.m.	3 31 4	☉̄ 11 43 45

The sun's compass bearing at the time of taking the first observation was S. by W., and the run of the ship in the interval between the observations was 30 miles on a S.W. by W. $\frac{3}{4}$ W. course. The chronometer was fast 27m. 19s. of G. M. T. on 17th November, and losing daily ·7s. The index correction of the sextant was — 1′ 17″, and height of the eye above the sea was 20 feet. Required the latitude and longitude of the place at the time of taking the second observation.

Ex. 470. 1887, September 1st, at ship, by reckoning in latitude 49° N., longitude 180° E., the following observations were made :—

App. time ship nearly.	*Time by chro.*	*Observed altitudes.*
0h. 33m. p.m.	0h. 42m. 57s.	☉̲ 48° 51′ 0″
4 41 p.m.	4 50 20	☉̄ 19 30 20

The sun's bearing at the first observation was S. by W., and the run of the ship in the interval was E.N.E, both by compass, at the rate of $6\frac{1}{2}$ knots per hour. The chronometer was fast 5m. 13s. on 12th May for G. M. T., and gaining daily 2s. The

height of the eye above the sea was 17 feet, and the index error of the sextant was + 1′ 10″. Find the ship's position at the second observation.

Ex. 471. 1887, March 20th, in latitude by account 26° 20′ S., longitude 115° 25′ W., the following observations were made:—

App. time ship nearly.	*Time by chro.*	*Observed altitudes.*
11h. 18m. a.m.	6h. 51m. 22s.	☉ 62° 7′ 20″
3 40 p.m.	11 5 10	☉ 30 31 40

The index correction of the sextant was + 3′ 24″, and height of eye above the sea 22 feet. The chronometer was slow of G. M. T. on 23rd February 23m. 39s., and losing daily 1·4s. The sun's bearing at the first observation by compass was S.E. by S., and the ship's course by compass was S.W. by W. during the interval, $7\frac{3}{4}$ knots per hour. Required the ship's position at the second observation.

Ex. 472. 1887, June 27. With a sextant whose index error was — 7′ 18″ at 15 feet above the level of the sea, the following observations were made in latitude by account 15° 10′ S., and longitude 78° 20′ E.:—

App. time ship nearly.	*Time by chro.*	*Observed altitudes.*
8h. 30m. a.m.	3h. 17m. 15s.	☉ 26° 50′ 10″
11 40 a.m.	6 26 1	☉ 51 6 15

The bearing of the sun at the first observation was S. 60° E., and course during the interval N.E., both by compass, at the rate of $7\frac{1}{2}$ knots per hour. The chronometer was slow on 28th April, 5m. 16s. of G. M. T., and gaining ·4s. daily. Required the position of the ship at the first observation.

Ex. 473. 1887, May 30th, at ship. The position of the ship by account was 7° 30′ N. latitude and 168° 40′ W. longitude, and the height of the eye above the sea was 24 feet. The first altitude was taken with a sextant whose index error was — 1′ 1″, and the second with one whose index error was + 3′ 20″.

App. time ship nearly.	*Time by chro.*	*Observed altitudes.*
0h. 24m. p.m.	11h. 32m. 52s.	☉ 74° 18′ 50″
4 20 p.m.	3 28 20	☉ 26 21 50

The chronometer on 3rd April was slow of G. M. T. 4m. 32s., and gaining daily ·7s. The sun's bearing at the latter observation was N. 73° W., and the ship's course in the interval between the observations was N.N.E., both by compass, at the

rate of $5\frac{3}{4}$ knots per hour. Find the ship's position at the second observation.

Ex. 474. 1887, November 29th, the following observations were made in latitude by account 6° 40′ S., longitude 179° 50′ E. :—

App. time ship nearly.	*Time by chro.*	*Observed altitudes.*
7h. 40m. a.m.	7h. 59m. 42s.	$\overline{\odot}$ 25° 44′ 45″
11 16 a.m.	11 34 30	$\underline{\odot}$ 71 32 45

The chronometer was on 19th September fast of G. M. T. 32m. 29s., and losing ·4s. daily. The index error of the sextant was + 1′ 44″, and height of the eye above the sea 21 feet. The compass bearing of the sun at the first observation was S. 60° E., and the compass course in the interval between the observations was S. 60° E., at the rate of $4\frac{3}{4}$ knots per hour. Required the position of the ship at the first observation.

Ex. 475. 1887, July 31st, at ship in latitude by account 30° 26′ N., longitude 132° 30′ E., the following observations were made :—

App. time ship nearly.	*Time by chro.*	*Observed altitudes.*
9h. 17m. p.m.	11h. 30m. 11s.	Vega 79° 1′ 40″
2 29 a.m.	4 41 58	Vega 33 38 50

The star at the second observation bore W.N.W., and the ship's course in the interval was due South, both by compass, at the rate of $6\frac{1}{2}$ knots per hour. The chronometer on 16th June was slow 1h. 2m. 53s. of G. M. T., and losing ·4s. daily. The index correction was — 2′ 29″, and height of the eye 23 feet. Find the latitude and longitude at the second observation.

SUMNER'S METHOD OF DOUBLE ALTITUDES.

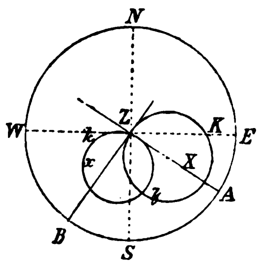
This method of finding latitude and longitude was propounded by Thomas B. Sumner, of Boston, Massachusetts, Captain in the United States Navy, about the year 1837, and made public in 1843. In the "Treatise on Navigation" it was stated that the position of any place on the earth was defined by the intersection of two lines, viz. by parallels of latitude and meridians of longitude. It is not necessary these should be the lines employed; for the intersection of *any* two lines through a point will define its exact position as well as those we have mentioned, provided the angle included between the lines be large enough to show clearly the point of intersection. Such is "Sumner's

Method;" he employed *lines of position* which we will endeavour to explain.

Circle of position and lines of position.—One-half of the earth is always turned towards any celestial object, so that a circle joining every place which has that object in the horizon, will be a great circle on the surface of the earth, whose pole is the point which has the object in its zenith. At all places situate on a small circle parallel to this great circle (and between it and the pole mentioned) the altitude of the body must be the same: such a circle is called a *circle of equal altitudes* or a *circle of position.* It is evident, then, if the altitude of an object be taken, the observer must be situated somewhere on a circle of equal altitudes. After a sufficient time has elapsed for the object to change its bearing as near eight points as convenient, let another altitude be taken, then it is plain that another circle of equal altitudes can be drawn through the observer's position. We have thus two circles, on both of which the ship must lie: hence she must be in the intersection of the circles; but as circles cut each other in two places, Ex. III. 10, she must be on one of the two points of intersection, and the latitude by account limits the position to one only of these points. Instead of employing the whole circles of position, Sumner used small portions only defined by the latitude by account, and in the calculations he considered them as straight lines, tangents at the point of intersection: these are the *lines of position.*

Illustration.—Let $N\ E\ S\ W$ be the plane of the observer's horizon, Z his zenith, X the position of the object at the first observation, $A\ X$ its altitude, and therefore $Z\ X$ its zenith distance. Then a circle drawn with X as pole, and at a distance $Z\ X$ will be a circle of position for the first altitude, because all points on the circle will be equally distant from that point which has X in the zenith. Let $Z\ K\ z$ be such a circle, then the observer must be situated somewhere on its circumference. When the object has moved to x let another circle of position $Z\ k\ z$ be drawn for the second altitude: then in a similar manner, the observer

Fig 53



must be on its circumference, and hence must be at one of the points of intersection, i.e. at either Z or z. In good observations, selected within proper limits, these points Z and z lie so far apart that with a very rough latitude by account there can be no difficulty in choosing the right one. Straight lines, which are tangents to $Z\,K\,z$ and $Z\,k\,z$ at the point Z, are the *lines of position.*

It is evident that if a great circle be drawn through the heavenly body under consideration and the zenith, it will cut the horizon at right angles, and its plane will cut every circle of position also at right angles: this great circle is a circle of altitude, or an azimuth circle, and hence *the bearing of the object is always at right angles to the line of position.*

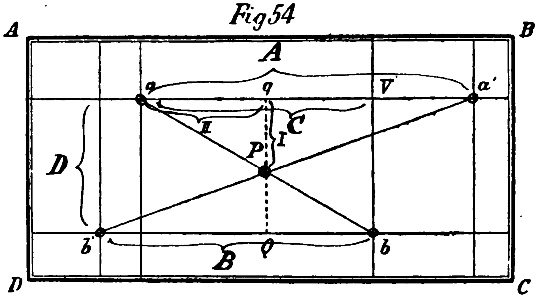
How the ship's position is determined.—It has been shown, p. 148, that a small error in latitude will produce a small error in the longitude deduced from a given latitude. Now, if we take a set of altitudes of any object, and assume any number of latitudes near each other, and not far from the true latitude, then calculate the longitude for each latitude respectively, and prick off on the chart the different positions thus found, we shall see they will all lie on or near a straight line which coincides with the line of position. As all the points lie so closely to this line, it is evident the same result will be obtained by assuming but two latitudes not far from the truth, and thus getting only two points on the chart: then, from what has already been said, the ship's position will lie on the line joining the two points. When the object has changed its bearing about eight points, with another set of altitudes, calculate and lay down on the same chart another line of position; this also passes through the place of observation, hence the intersection of the two lines on the chart will give the ship's position.

If a chart cannot be obtained on a scale sufficiently large, the portion of the globe which is wanted should be projected as directed when treating of Mercator's charts in the "Treatise on Navigation." The following is a projection of a small one for our purpose.

Let $A\;B\;C\;D$ be the portion of the earth required. With the first altitude and *two assumed latitudes* not far from the latitude by account, calculate the longitude of the ship for each latitude: let a and b be the positions on the chart deduced from these calculations. Join a and b, this is the line of position for the first altitude. With the second altitude and

the same latitudes again calculate the longitudes, and let a' and b' be the positions given by the second observation. Join a' and b', this is the line of position for the second altitude.

Fig 54



The point of intersection P is the place of the ship. Draw $q\,P\,Q$ a meridian through P, then $P\,Q$ is the correction for latitude reckoned from $b\,b'$, and $P\,q$ the correction reckoned from $a\,a'$.

Let the difference of assumed latitudes $q\,Q = \mathrm{D}$;
the less difference of longitude given by either latitude $b\,b' = B$;
the greater difference of longitude given by either latitude $a\,a' = A$;
the difference of longitude between a and b or $a\,V = C$.

Then $\because$ the triangles $a\,P\,a'$, $b\,P\,b'$ are similar.

$$\therefore \frac{b\,b'}{a\,a'} = \frac{P\,Q}{P\,q} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a);$$

$$\frac{b\,b' + a\,a'}{a\,a'} = \frac{P\,Q + P\,q}{P\,q} = \frac{Q\,q}{P\,q};$$

$$\text{or } \frac{A + B}{A} = \frac{D}{P\,q}.$$

Hence $P\,q$ or the correction for latitude of the parallel on which is the greatest difference of longitude $= \dfrac{A \times D}{A + B}$ . . . I.

Again, by similar triangles, $a\,P\,q$, $a\,V\,b$,

$$\frac{a\,q}{P\,q} = \frac{a\,V}{b\,V};$$

$$\therefore a\,q = \frac{a\,V \times P\,q}{b\,V}.$$

Substituting the value of $P\,q$ found above in I., we get—

$$a\,q = \frac{C \times A \times D}{D(A+B)}$$
$$= \frac{A \times C}{A+B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{II.}$$

Hence correction for the longitude $a = \dfrac{A \times C}{A+B}$.

The lines of position will not always cut one another *between* the assumed parallels of latitude; they must then be produced to meet. The point of intersection will still be the position of the ship, and the same formulæ as above, with the exception that B will be negative, may be obtained by subtracting unity from each side of equation (a) instead of adding it.

$$\left.\begin{aligned}\text{Hence correction for latitude} &= \frac{A \times D}{A \pm B}\\ \text{and correction for longitude} &= \frac{A \times C}{A \pm B}\end{aligned}\right\}.$$

RULE FOR SUMNER'S METHOD.

RULE (a) Find the Greenwich time for each observation.

(b) Find the declinations, polar distances, and equations of time corresponding to the Greenwich dates.

(c) Obtain the correction for run during the interval, and correct the altitude for each observation.

(d) Assume two latitudes differing about a degree or less, and not widely different from the latitude by dead reckoning. Neither of the two assumed latitudes should be as great as the sum of the declination and the zenith distance when these are of the same name; nor as great as the difference between the declination and the zenith distance when these are of different names. Because, if either latitude be assumed too great, the altitude of the object would be greater than the meridian altitude for that day, which is impossible.

(e) Compute four longitudes thus:—

(1) With the *first* altitude the less assumed latitude and polar distance corresponding to the time of the first altitude: call this a.

(2) With the *first* altitude the greater assumed

latitude and polar distance corresponding to the time of the first altitude: call this b.

(3) With the *second* altitude the less assumed latitude and polar distance corresponding to the time of the second altitude: call this a'.

(4) With the *second* altitude the greater assumed latitude and polar distance corresponding to the time of the second altitude: call this b'.

(f) Then form a table similar to the annexed.

Diff. of lat. = D.	With the less lat.		With the greater lat.	
1st observation *longitude*	a		b	
2nd observation *longitude*	a'		b'	
	Diff. long.		Diff. long.	

The difference of longitudes are marked E. or W., according as the longitudes at the second observation are east or west of those at the first observation for the same latitude. The greater difference of longitude we call A, the less B. If both the differences of longitude be of the same name, their difference must be employed ($A - B$); but if one be east and the other west, their sum must be used ($A + B$).

The latitudes which stand over the greater difference of longitude A should be corrected, although either may be used. The correction is computed thus:—Multiply the greater difference of longitude A by the difference of assumed latitudes D, and divide the product by the sum or difference of longitudes, according as they are of different or of the same name.

$$\text{Cor. for lat.} = \frac{A \times D}{A \pm B}.$$

When found, the correction must be applied towards the latitude not corrected.

(g) The correction for longitude is thus found:—Find the difference of longitude given by using both latitudes and the altitude at the first observation $(a \backsim b) = C$. Multiply this difference of longitude C by the

greater difference of longitude A, and divide by the sum or difference of the differences of longitudes similar to the above.

$$\text{Cor. for long.} = \frac{A \times C}{A \pm B}.$$

When found, the correction must be applied to the longitude at the first altitude which stands over the greater difference of longitude A and towards the other used in finding C.

Ex. 476. 1887, February 28th, in latitude by account 47° 20′ N., longitude 133° 10′ W., the following observations were made:—

App. time ship nearly.	*Chronometer.*	*Observed altitudes.*
9h. 53m. a.m.	6h. 32m. 3s.	☉ 27° 55′ 0″
0 56 p m.	9 35 15	☉ 33 32 0

The sun's magnetic bearing at the first observation was S.E. by E. ¼ E., and the course of the ship in the interval was magnetic S. by E. ¾ E., at the rate of 6 miles per hour. The chronometer, was slow of G. M. T. on 4th January, 27m. 10s., and gaining daily ·6s. The index error of the sextant was − 2′ 19″, and the height of the eye 23 feet. Required the latitude and longitude of the ship at the time of taking the less altitude, assuming the latitudes to be 47° 10′ N. and 47° 40′ N.

For Greenwich apparent time nearly.

	First observation.		*Second observation.*
February	27d. 21h. 53m. 0s.		Feb. 28d. 0h. 56m. 0s.
Long. in time	+ 8 52 40		8 52 40
Greenwich app. time	28 6 45 40	Feb. 28	9 48 40

For correct Greenwich mean time.

	First observation.		*Second observation.*
Time by chronometer	6h. 32m. 3s.		9h. 35m. 15s.
Slow	+ 27 10		+ 27 10
	6 59 13		10 2 25
Gain 55·29 × ·6	− 33·17	Gain 55·42 × ·6	− 33·25
G. M. T. Feb. 28	6 58 39·83	Feb. 28	10 1 51·75

For P. D., first observation.

Sun's declination	7° 58′ 6·0″ S.
Correction	— 6 35·7
True declination	7 51 30·3 S.
N. P. D.	97 51 30 3

Variation of declination.

In 1 hour	— 56·69″
No. hrs.	6 98
	60)395·6962
Correction	— 6′ 35·7″

For P. D., second observation.

Sun's declination	7° 58′ 6·0″ S.
Correction	— 9 28·7
True declination	7 48 30·3 S.
N. P. D.	97 48 37·3

Variation of declination.

In 1 hour	— 56·69″
No. hrs.	10·031
	60)568·65739
Correction	9′ 28·7″

For eq. time, first observation.

Equation time	+ 12m. 45·05s.
Correction	— 3·28
True equation time	+ 12 41·77

Variation of eq. time.

In 1 hour	— ·470s.
No. hrs.	6·98
Correction	3·28080

For eq. time, second observation.

Equation time	+ 12m. 45·05s.
Correction	— 4·71
True equation time	+ 12 40·34

Variation of eq. time.

In 1 hour	— ·470s.
No. hrs.	10·031
Correction	4·714570

Correction for run.

Sun's mag. bearing	S. $5\frac{1}{4}$ E.
Ship's course	S. $1\frac{3}{4}$ E.
Included angle	$3\frac{1}{2}$ pts.

Included angle $3\frac{1}{2}$ pts.	cos. 9·888185
Run 3·053 × 6· = 18·322	log. 1·262925
Cor. for run 14·16′	log. 1·151110

For alt. first observation.

Observed alt. L. L.	27° 55′ 0″
Index error	— 2 19
	27 52 41
Dip	— 4 43
	27 47 58
Semidiameter	+ 16 10
	28 4 8
Ref. and parallax	— 1 38
True altitude	28 2 30

For alt. second observation.

Observed alt. L. L.	33° 32′ 0″
Index error	— 2 19
	33 29 41
Dip	— 4 43
	33 24 58
Semidiameter	+ 16 10
	33 41 8
Ref. and parallax	— 1 18
True altitude	33 39 50
Correction for run	— 14 10
Reduced alt.	33 25 40

For long. a with lat. 47° 10′ *N.*

a	28°	2′	30″		
l	47	10	0	sec.	·167575
p	97	51	30	cosec.	·004098
	173	4	0		
S	86	32	0	cos.	8·781524
$S-a$	58	29	30	sin.	9·930727
					2)18·883924
$\frac{h}{2}$	16°	3′	41·6″	sin.	9·441962
			2		
h	32	7	23·2		
			4		
	60)128	29	32·8		
Eastern hr. angle	2h.	8m.	29·55s.		
Western hr. angle	21	51	30·45		
Equation time	+	12	41·77		
M.T. ship, Feb. 27	22	4	12·22		
M. T. G., Feb. 28	6	58	39·83		
	8	54	27·01		
Longitude a	133°	36′	54″ W.		

For long. b with lat. 47° 40′ *N.*

a	28°	2′	30″		
l	47	40	0	sec.	·171699
p	97	51	30	cosec.	·004098
	173	34	0		
S	86	47	0	cos.	8·749055
$S-a$	58	44	30	sin.	9·931883
					2)18·856735
$\frac{h}{2}$	15°	33′	13·8″	sin.	9·428367
			2		
h	31	6	27·6		
			4		
	60)124	25	50·4		
Eastern hr. angle	2h.	4m.	25·82s.		
Western hr. angle	21	55	34·18		
Equation time	+	12	41·77		
M.T. ship, Feb. 27	22	8	15·95		
M. T. G., Feb. 28	6	58	39·83		
	8	50	23·88		
Longitude b	132°	35′	58″ W.		

For long. a′ with lat. 47° 10′ *N.*

a	33°	25′	40″		
l	47	10	0	sec.	·167575
p	97	48	37·3	cosec.	·004048
	2)178	24	17·3		
S	89	12	8·65	cos.	8·143635
$S-a$	55	46	29	sin.	9·917417
					2)18·232675
$\frac{h}{2}$	7°	30′	40	sin.	9·116337
			2		
h	15	1	20		
			4		
	60	5	20		

For long. b′ with lat. 47° 40′ *N.*

a	33°	25′	40″		
l	47	40	0	sec.	·171699
p	97	48	37·3	cosec.	·004048
	2)178	54	17·3		
S	89	27	8·65	cos.	7·980306
$S-a$	56	1	29	sin.	9·918700
					2)18·074753
$\frac{h}{2}$	6°	15′	25″	sin.	9·037376
			2		
	12	30	50		
			4		
	50	3	20		

Western hr. angle	1h.	0m.	5·33s.	Western hr. angle	0h.	50m.	3·33s.
Equation time	+	12	40·34	Equation time	+	12	40·34
M. T. ship, Feb. 28	1	12	45·67	M. T. ship, Feb. 28	1	2	43·67
M. T. G., Feb. 28	10	1	51·75	M. T. G., Feb. 28	10	1	51·75
	8	49	6·08		8	59	8·08
Longitude a'	132° 16′ 31″ W.			Longitude b'	134° 47′ 1″ W.		

Diff. lat. $D = 30'$.	With lat. 47° 10′ N.		With lat. 47° 40′ N.	
First obser. *longitude*	a	133° 36′ 54″ W.	b	132° 35′ 58″ W.
Second obser. *longitude*	a'	132 16 31 W.	b'	134 47 1 W.

$B = 1\ 20\ 23$ E. $\quad A = 2\ 11\ 3$ W.
$B = 1\ 20\ 23$ E.
$A + B = 3\ 31\ 26$

$$\text{Correction for lat.} = \frac{A \times D}{A + B} = \frac{2^\circ\ 11'\ 3'' \times 30'}{3^\circ\ 31'\ 26''} = 18'\ 36''$$

a	=	133° 36′ 54″ W.
b	=	132 35 58 W.
C	=	1 0 56 W.

$$\text{Correction for long.} = \frac{A \times C}{A + B} = \frac{2^\circ\ 11'3'' \times 1^\circ\ 0'\ 56''}{3^\circ\ 31'\ 26''} = 37'\ 46''\ \text{W.}$$

Assumed latitude	47° 40′ 0″ N.
Correction for lat.	— 18·36
True latitude	47 21 24 N.

Long. to be cor.	132 35′ 58″ W.
Cor. for long.	37 46 W.
True longitude	133 13 44 W.

By comparing the latitude and longitude obtained by Sumner's method with the results given by the direct method, and also with those by Ivory's method, it will be seen how closely these approximate to one another, and to the true latitude 47° 22′ N. and longitude 133° 13′ W. with which the question was formed.

PROJECTION OF THE QUESTION.—The following is a projection of the question on the scale of one degree to an inch and a half.

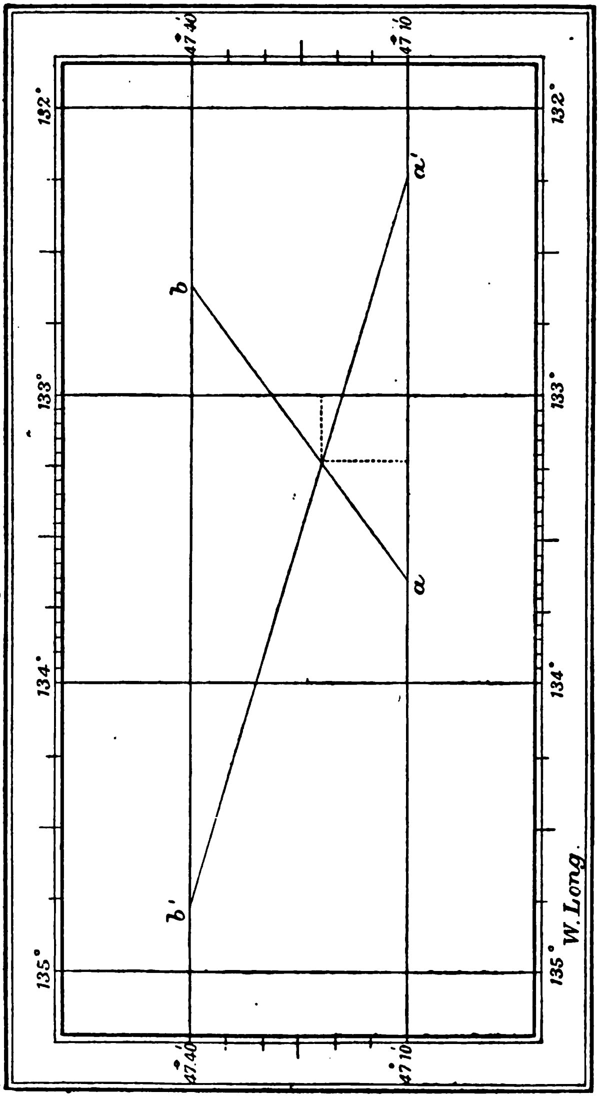


Fig 55

By projection to this scale we find the latitude in 47° 22′ N. and longitude in 133° 12½′ W.

The latitude found by Sumner's method is independent of the error of the time-keeper; but longitude cannot be found with precision unless we know both the error and the rate of the watch with accuracy. The error of the chronometer would place the line of position too far to the east or to the west of its true position, but it would not alter its direction.

Advantages of Sumner's method.—To fetch a small island it is customary to sail into the parallel of latitude on which the island is situated, and then run down the longitude until the island is reached. This process may be avoided by laying down the line of position on the chart, and if it pass through the island the vessel should be steered on that course until it is reached; but if the line of position does not pass through the required point, a line parallel to the line of position should be drawn on the chart through the island, and the vessel steered until she arrives to this parallel line, and then kept on it until the object be attained. Thus the direction of the line of position furnishes us with data by which the intended port may be reached with the greatest certainty, although neither the latitude nor the longitude is known. It should also be noticed the line can be laid down even if the state of the weather should prevent a second altitude by which the position of the ship may be found.

When, from unavoidable circumstances, only one line of position can be laid on the chart, the course and distance the ship has made good from the time of taking the observation up to noon should also be laid down on the same chart, and a line parallel to the line of position be drawn through the position of the ship. This will be the line of position at noon, and the vessel will be situated somewhere on this line. Now, if the latitude by meridian altitude can be determined, the point in the line of position at noon corresponding to the latitude will also give the longitude of the ship at noon. Or, again, if the line of position be sailed on until a light be sighted, the bearing of the light by compass must be reduced to the proper bearing and laid on the chart. Where these lines intersect will determine the place of the ship.

In the foregoing remarks we have supposed the chronometer is known to be correct; but even if the chronometer cannot be

relied on, the line of position may be utilized as follows. Now, the line of position from a single observation is always *correct in direction,* but may be too far east or too far west of its true place. We must therefore draw several lines on the chart having the correct bearing; then, by sailing in the direction of a line of position, and taking frequent casts of the lead, and comparing the depths thus obtained with those marked on the chart, the particular line of position on which the ship has been sailing can be selected and the actual place of the ship determined.

A single line of position may also be utilized in finding the error of the compass. This is thus done. Calculate the direction of the line of position by either middle latitude sailing or by Mercator's sailing, using as data the two assumed latitudes and the corresponding longitudes, or else by means of a protractor take its direction from the chart. Then, because the line of position is a tangent to the circle of position at the place the observer is situated, therefore the bearing of the object is at right angles to the line of position, and hence its true bearing is at once deduced. As the azimuth of the object is observed at the same time as the altitude for the correction for run, very little additional labour is entailed in finding the error of the compass; and correct results are obtained, although neither the latitude nor longitude are known within several miles.

This is illustrated by using the data already found in the question at the first observation.

a	lat. 47° 10′ N.		long. 133° 36′ 54″ W.
b	lat. 47 40 N.		long. 132 35 58 W.
	Diff. lat 30 N.	Diff. long.	60 93 E.

Mid latitude 47° 25′ N.

$$\text{Tan. direction line of position} = \frac{d\,.\,\text{long.} \times \cos.\ \text{mid. lat.}}{d\,.\,\text{lat.}}$$

$$= \frac{60.93 \times \cos.\ 47°25'}{30}$$

$$= \tan.\ 53°\ 57\tfrac{1}{2}'.$$

$\therefore$ Direction of line of position is N. 53° 57½′ E.

Hence the bearing of the sun at the first observation was S. 36° 2½′ E.

This may be verified by computing the true azimuth from the data found in the question.

p	97°	51′	30″		
l	47	22	0	sec.	·169216
a	28	2	30	sec.	·054234
	2)173	16	0		
S	86	38	0	cos.	8·768828
$S \backsim p$	11	13	30	cos.	9·991611
					2)18·983889

Half azimuth	S. 18° 5′ E.	sin.	9·491944
True azimuth	S. 36 10 E.		

thus agreeing almost exactly with that obtained by mid-lat. sailing. The variation is then found thus :—

Magnetic bearing of sun	S. 59° 4′ E.
True " "	S. 36 10 E.
Variation	22 54 E.

It has already been shown that the point vertically under the sun is the centre of the circle of position. Its bearing is therefore S. 36° 10′ E., and its distance will be the zenith distance of the sun, viz. 61° 57′ 30″, or 3717½ miles.

It has been proved, pp. 147 to 151, that the altitudes for time should be taken when the object is as far from the meridian as practicable ; but we also stated, when treating of "*the most advantageous position of objects for double altitudes*," the greater altitude should be taken as near noon as possible ; hence what would be favourable positions of objects for determining latitude and longitude by either the direct or Ivory's method, would be unfavourable for Sumner's method. This is another reason why a discrepancy often exists in the results when working the same question by the three different methods. If the hour angle, when least, be from one to two hours, the answers by all three methods should agree very nearly. The student should compare the results obtained in the following questions with those already found by the *direct method* and *Ivory's method.*

EXERCISE XXI.

Ex. 477. 1887. On December 24th, in latitude by account

47° 30′ N., longitude 8° 30′ W., the following observations were made:—

App. time ship nearly.	*Time by chro.*	*Observed altitudes.*
11h. 15m. a.m.	0h. 16m. 6s.	$\overline{\odot}$ 18° 21′ 20″
2 30 p.m.	3 31 4	$\overline{\odot}$ 11 43 45

The sun's compass bearing at the time of taking the first observation was S. by W., and the course of the ship in the interval between the observations was 30 miles on a S.W. by W. $\frac{3}{4}$ W. course. The chronometer was fast 27m. 19s. of G. M. T. on 17th November, and losing ·7s. daily. The index correction of the sextant was − 1′ 17″, and height of the eye above the sea was 20 feet. Required the latitude and longitude of the place where the second observation was made, assuming the latitudes to be 47° 20′ N. and 48° N.

Ex. 478. 1887, September 1st, at ship, by reckoning in latitude 49° N., longitude 180° E., the following observations were made:—

App. time ship nearly.	*Time by chro.*	*Observed altitudes.*
0h. 33m. p.m.	0h. 42m. 57s.	$\underline{\odot}$ 48° 51′ 0″
4 41 p.m.	4 50 20	$\overline{\odot}$ 19 30 20

The sun's bearing at the first observation was S. by W., and the course of the ship in the interval was E.N.E., both by compass, at the rate of $6\frac{1}{2}$ knots per hour. The chronometer was fast 5m. 13s. on 12th May for G. M. T., and gaining 2s. daily. The height of the eye above the sea was 17 feet, and the index error of the sextant was + 1′ 10″. Find the ship's position at the second observation, assuming the latitudes to be 48° 30′ N. and 49° N.

Ex. 479. 1887, March 20th, in latitude by account 26° 20′ S., longitude 115° 25′ W., the following observations were made:—

App. time ship nearly.	*Time by chro.*	*Observed altitudes.*
11h. 18m. a.m.	6h. 51m. 22s.	$\underline{\odot}$ 62° 7′ 20″
3 40 p.m.	11 5 10	$\underline{\odot}$ 30 31 40

The index correction of the sextant was + 3′ 24″, and height of the eye above the sea 22 feet. The chronometer was slow of G. M. T. on 23rd February, 23m. 39s., and losing daily 1·4s. The sun's bearing at the first observation was S.E. by S. by compass, and the ship's course by the same was S.W. by W. during the interval $7\frac{3}{4}$ knots per hour. Required the ship's position at the second observation, assuming the latitudes to be 26° S. and 26° 20′ S.

Ex. 480. 1887, June 27th. With a sextant whose index error was — 7′ 18″ at 15 feet above the sea level, the following observations were made in latitude by account 15° 10′ S., and longitude 78° 20′ E.

App. time ship nearly.	*Time by chro.*	*Observed altitudes.*
8h. 30m. a.m.	3h. 17m. 15s.	☉̲ 26° 50′ 10″
11 40 a.m.	6 26 1	☉̅ 51 6 15

The bearing of the sun at the first observation was S. 60° E., and course during the interval N.E., both by compass, at the rate of $7\frac{1}{2}$ knots per hour. The chronometer was slow on 28th April, 5m. 16s. of G. M. T., and gaining ·4s. daily. What was the latitude and longitude of the ship, and the deviation of the compass at the first observation, if the variation at the place was 4° 50′ W.? Assume latitudes 14° 40′ S. and 15° 10′ S.

Ex. 481. 1887, May 30th, at ship. The position of the ship by account was 7° 30′ N. latitude, and 168° 40′ W. longitude, and height of the eye above the sea was 24 feet. The first altitude was taken with a sextant whose index error was — 1′ 1″, and the second with one whose index error was + 3′ 20″.

App. time ship nearly.	*Time by chro.*	*Observed altitudes.*
0h. 24m. p.m.	11h. 32m. 52s.	☉̅ 74° 18′ 50″
4 20 p.m.	3 28 20	☉̲ 26 21 50

The chronometer on 3rd April was slow of G. M. T. 4m. 32s., and gaining daily ·7s. The sun's bearing at the latter observation was N. 73° W., and the ship's course in the interval between the observations was N.N.E., both by compass, at the rate of $5\frac{3}{4}$ knots per hour. Find the ship's position, the error of the compass, and the course and distance to the centre of the circle of position at the second observation. Assume latitudes 7° 30′ N. and 7° 50′ N.

Ex. 482. 1887, November 29th, the following observations were made in latitude by account 6° 40′ S., longitude 179° 50′ E. :—

App. time ship nearly.	*Time by chro.*	*Observed altitudes.*
7h. 40m. a.m.	7h. 59m. 42s.	☉̲ 25° 44′ 45″
11 16 a.m.	11 34 30	☉̅ 71 32 45

On 19th September the chronometer was 32m. 29s. fast of G. M. T., and losing ·4s. daily. The index error of the sextant was + 1′ 44″, and height of the eye above the sea 21 feet.

The compass bearing of the sun at the first observation was S. 60° E., and the compass course in the interval between the observations was S. 60° E., at the rate of $4\frac{3}{4}$ knots per hour. Find the latitude and longitude of the ship at first observation, the deviation of the compass if the variation at the place be 9° 50′ E., and the course and distance to the centre of the first circle of position. Assume latitudes 6° S. and 6° 50′ S

Ex. 483. 1887, July 31st, at ship, in latitude by account 30° 26′ N., longitude 132° 30′ E., the following observations were made :—

App. time ship nearly.	*Time by chro.*	*Observed alts. Vega.*
9h. 17m. p.m.	11h. 30m. 11s.	79° 1′ 40″
2 29 a.m.	4 41 58	33 38 50

The star at the second observation bore W.N.W., and the ship's course in the interval was due South, both by compass, at the rate of $6\frac{1}{2}$ knots per hour. On 16th June the chronometer was slow 1h. 2m. 53s. of G. M. T., and losing ·4s. daily. The index correction of the sextant was — 2′ 29″, and height of the eye 23 feet. What was the latitude and longitude of the ship at the second observation? Also the compass error at that time, and the course and distance to the centre of the second circle of position? Assume latitudes 30° 10′ N. and 30° 40′ N.

Ex. 484. 1887, February 14th, p.m., the position of the ship by account was 50° 20′ N. latitude, and 39° 20′ W. longitude. The chronometer showed 0h. 54m. 22s., and was slow 40m. 51s. on 13th January, and losing 3·5s. daily. The altitude of Procyon W. of the meridian was observed to be 43° 12′ 20″, index error — 2′ 11″. At the same time, by another observer, the altitude of Arcturus E. of the meridian was 19° 36′; index error + 1′ 29″, height of the eye 21 feet. If the variation of the compass at the place be 35° 15′ W., find the deviation of the compass for the then position of the ship's head, the latitude and longitude of the ship, and the bearing and distance of the centre of the circle of position calculated from Procyon, if its compass bearing be S.W. by S. $\frac{1}{4}$ S., assuming the latitudes to be 50° 10′ N. and 50° 40′ N.

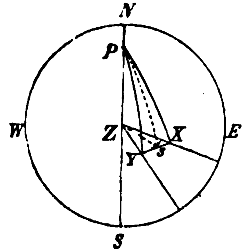
Ex. 485 1887, April 8th, a.m., in latitude by account 45° 40′ S., longitude 68° 40′ E., the observed altitude of Spica W. of the meridian was 13° 7′ 0″ at about 5h. 45m. a.m. apparent time at ship: and about 5h. 46m. 40s. the observed altitude of

Altair E. of the meridian was 34° 23′ 0″. The chronometer at the latter observation showed 12h. 4m. 7s., which on the 26th February had been found 1h. 9m. 14s. slow of G. M. T., and gaining 3·2s. daily. Index error + 1′ 14″, height of the eye 25 feet. Required the position of the ship; the error of the compass computed from Spica, if its compass bearing be West: and the bearing and distance of the centre of the circle of position given by Altair. Assume latitudes 45° 45′ S. and 46° 0′ S.

THE SHORT METHOD OF DOUBLE ALTITUDES.

This method is so called because the observations must be made in quick succession, and hence the change in the declination of an object may be neglected. It will be seen in the following investigation that, like other methods of double altitudes, the correct time at ship is not required, but the interval between the observations must be known with accuracy.

Fig 56



Let a_1, z_1, and h_1 be the altitude, zenith distance, and hour angle of the object at X;
a_2, z_2, and h_2 be the altitude, zenith distance, and hour angle of the object at Y;
p and l' the polar distance and colatitude. These are assumed constant. Then

$$\cos. z_1 = \cos. p . \cos. l' + \sin. p . \sin. l' . \cos. h_1$$
$$\cos. z_2 = \cos. p . \cos. l' + \sin. p . \sin. l' . \cos. h_2 .$$

Hence $\cos. z_2 - \cos. z_1 = \sin. p . \sin. l' (\cos. h_2 - \cos. h_1)$, and

$$2 \sin. \frac{z_1 - z_2}{2} . \sin. \frac{z_1 + z_2}{2} = \sin. p . \sin. l' . 2 \sin. \frac{h_1 - h_2}{2} . \sin. \frac{h_1 + h_2}{2} \quad \text{I.}$$

Now, as the interval between the observations is supposed short,—

$$\frac{z_1 + z_2}{2} = z, \text{ the zenith distance at the mid time;}$$

$$\frac{h_1 + h_2}{2} = h, \text{ the hour angle at the mid time;}$$

$$\sin. \frac{z_1 - z_2}{2} = \frac{z_1 - z_2}{2} . \sin. 1'';$$

$$\text{and } \sin. \frac{h_1 - h_2}{2} = \frac{h_1 - h_2}{2} . \sin. 1''.$$

Making these substitutions, Equation I. becomes

$$(z_1 - z_2) \sin. z = (h_1 - h_2) \sin. p \,.\, \sin. l' \,.\, \sin. h \,;$$

$$\therefore \frac{\sin. l' \,.\, \sin. h}{\sin. z} = \frac{z_1 - z_2}{h_1 - h_2} \,.\, \text{cosec.}\, p \quad . \quad . \quad . \quad . \quad \text{II.}$$

Now, if s be the position of the object at the mid time, from the triangle $P\,s\,Z$ we get

$$\sin. P\,s\,Z = \frac{\sin. l' \,.\, \sin. h}{\sin. z} \,;$$

$$\therefore \sin. P\,s\,Z = \frac{a_2 - a_1}{h_1 - h_2} \,.\, \text{cosec.}\, p \quad . \quad . \quad \text{III.}$$

This angle $P\ s\ Z$ is sometimes called the angle of position, which, having found, we have p. z, and the included angle $P\,s\,Z$ to find the colatitude $P\,Z$. Then

$$\cos. l' = \cos. p \,.\, \cos. z + \sin. p \,.\, \sin. z \,.\, \cos. P\,s\,Z$$

$$= \cos. p \,.\, \sin. a + \sin. p \,.\, \cos. a \left(1 - 2 \sin.^2 \frac{P\,s\,Z}{2}\right)$$

$$= \sin. (p + a) - 2 \sin. p \,.\, \cos. a \,.\, \sin.^2 \frac{P\,s\,Z}{2}.$$

In this formula we know $\cos. l'$ and sine $(p + a)$ must be proper fractions; hence also $2 \sin. p \,.\, \cos. a \,.\, \sin.^2 \frac{P\,s\,Z}{2}$ must also be a proper fraction. Let it therefore $= \sin. \theta$; then

$$\cos. l' = \sin. (p + a) - \sin. \theta$$

$$\sin. l = 2 \sin. \frac{p + a - \theta}{2} \,.\, \cos. \frac{p + a + \theta}{2} \quad . \quad . \quad \text{IV.}$$

This completes the investigation, and from it the following rule is derived :—

RULE (a) Find the interval between the observations $(h_2 \backsim h_1)$, reduce it to seconds of time, and multiply the result by 15.

(b) Reduce the mid time of observation to Greenwich time.

(c) Correct the declination for the Greenwich date, and find the polar distance.

(d) Correct the altitudes, find their difference, and reduce it to seconds $(a_2 - a_1)''$.

(e) Find half the sum of the altitudes a, to which add the polar distance $(p + a)$.

(f) Add together—

log. diff. alts. . . . log. $(a_2 - a_1)$
log. cosec. polar dist. log. cosec. p,

and from the result subtract log. of the interval, reduced to seconds of arc log. $(h_1 - h_2)''$; the result is log. sine of the angle of position.

$$\text{Sin. } P s Z = \frac{a_2 - a_1}{h_1 - h_2} \text{. cosec. } p.$$

(*g*) Add together—

Twice log. sine of half the angle of position .	log. $\sin.^2 \frac{P s Z}{2}$
log. sine polar distance	log. sin. p
log cos. mean of altitudes	log. cos. a
log. 2	log. 2.

The result is log. sin. θ.

$$\sin. \theta = 2 \sin. p \,.\, \cos. a \,.\, \sin.^2 \frac{P s Z}{2}.$$

(*h*) Take half the difference and half the sum of $(p + a)$ found in (*e*) and θ found in (*g*); then add together log. 2, log. sin. $\frac{p + a - \theta}{2}$ and log. cos. $\frac{p + a + \theta}{2}$.

The result is log. sin. latitude.

$$\sin. l = 2 \cos. \frac{p + a + \theta}{2} \,.\, \sin. \frac{p + a - \theta}{2}.$$

Ex. 486. 1887, May 25th, at 9h. 27m. 15s. a.m. apparent time at ship, in longitude 41° 15′ W., the observed altitude of the sun's lower limb was 51°, and at 9h. 38m. 20s. a.m. the observed altitude of the lower limb was 52° 49′ 50″. Index correction for sextant — 2′ 4″, height of eye 19 feet. Required the latitude.

For mid time at Greenwich.

	h.	m.	s.
First observation, May 24	21h.	27m.	15s.
Second observation	21	38	20
	2)43	5	35
Mid time ship, May 24	21	32	47·5
Long., W.	+ 2	45	0
Mid time Greenwich, May 25	0	17	47·5

For elapsed time.

. .	21h. 27m.	15s.
. .	21 38	20
Interval	11	5
	60	
	665s.	
	15	
$h_2 \backsim h_1$	9975″	

For polar distance.

Declination May 25	20° 56′ 36·9″ N.
Correction	+ 8·1
True declination	20 56 45 N.
N. P. D.	69 3 15

Variation of declination.

In 1 hour	+ 26·99″
No. hours	·3
Correction	8·097

To correct first altitude.

Observed alt. L. L.	51° 0′ 0″
Index error	− 2 4
	50 57 56
Dip	− 4 17
	50 53 39
Semidiameter	+ 15 49
	51 9 28
Refraction	− 46
	51 8 42
Parallax	+ 5
1st true altitude	51 8 47
2nd "	52 58 40
	2)104 7 27
a	52 3 43·5
p	69 3 15
$p + a$	121 6 58·5

To correct second altitude.

Observed alt. L. L.	52° 49′ 50″
Index error	− 2 4
	52 47 46
Dip	− 4 17
	52 43 29
Semidiameter	+ 15 49
	52 59 18
Refraction	− 43
	52 58 35
Parallax	+ 5
2nd true altitude	52 58 40
1st "	51 8 47
Difference	1 49 53
	60
	109
	60
$a_1 \backsim a_2$	6593″

To find angle of position.

p	69° 3′ 15″	cosec.	·029691
$a_1 \backsim a_2$	6593	log.	3·819083
			3·848774
$h_1 \backsim h_2$	9975	log.	3·998913
Angle of position	45 2 58·6	sin.	9·849861

$\therefore \frac{PXZ}{2}$ 22 31 29·3

$\frac{p + a + \theta}{2} = 65° 24' 33''$

To find angle θ.

$\frac{PXZ}{2}$	22° 31′ 29·3″	sin.	9·583294
			2
			19·166588
p	69 3 15	sin.	9·970309
a	52 3 43·5	cos.	9·788739
	2	log.	·301030
θ	9° 42′ 7·5″	sin.	9·226666

$p + a$ 121 6 58·5

$\frac{p + a - \theta}{2} = 55° 42' 25·5''$

To compute the latitude.

$\frac{p + a + \theta}{2}$	65° 24′ 33″	cos.	9·619234
$\frac{p + a - \theta}{2}$	55 42 25·5	sin.	9·917069
2		log.	·301030
Latitude	43 26 24	sin.	9·837333

This method cannot be recommended for sea purposes, nor when others are available, because a small error in either altitude will cause a large error in the latitude; but on shore, where a mean of several observations taken in an artificial horizon can be obtained, the results are fairly correct.

John Douwes of Amsterdam, about the year 1740 proposed another method for the solution of the double altitude problem. He published tables for the use of the Dutch captains; but Dr. Pemberton, in 1760, gave in the Philosophical Transactions the demonstration for the construction of these tables. Douwe's method is somewhat similar to the short method; but it is varied by finding sin. $\frac{h_1 + h_2}{2}$ when the observations are taken on the same side of the meridian, and sin. $\frac{h_1 - h_2}{2}$ when taken on opposite sides of the meridian from Equation I., using the latitude by account. Then

$$\sin.\frac{h_1 + h_2}{2} = \sin.\frac{a_2 + a_1}{2}.\sin.\frac{a_2 - a_1}{2}.\text{cosec.}\frac{h_1 - h_2}{2}.\text{cosec.}p.\sec.l$$

$$= \frac{a_2 - a_1}{h_1 - h_2}.\sin.\frac{a_2 + a_1}{2}\ \text{cosec.}\ p\ .\ \sec.\ l.$$

Thus $\frac{h_1 + h_2}{2}$ is found, and $\frac{h_1 - h_2}{2}$ is known to be half the elapsed time; hence the hour angle at the time of each observation is known, and the question is completed as a reduction to the meridian.

Exercise XXII.

Ex. 487. 1887, July 17th, in latitude by account 29° 30′ S., longitude 63° 49′ 15″ E., at 2h. 52m. 35s. p.m. apparent time at ship, the sun's lower limb had an altitude of 24° 22′ 30″, and at 3h. 1m. 8s. p.m., by the same watch, the altitude of the lower limb was 23° 3′ 15″. Index error—2′ 25″, height of eye 22 feet. Required the latitude.

Ex. 488. 1887, June 22nd, in latitude by account 46° N., longitude 166° 22′ E., the means of observed altitudes and apparent times were as follows:—First observation: apparent time at ship 8h. 26m. 20s. a.m., altitude sun's L. L. 41° 51′ 51″. Second observation: apparent time 8h. 36m. 44s. a.m., altitude sun's L. L. 43° 42′ 59″. Index error + 1′ 14″, height of eye 21 feet. Find the latitude.

Ex. 489. 1887, November 5th, in longitude 57° 20′ W., the means of observed altitudes and apparent times were as follows:—First observation, sun's L. L. 32° 54′ 31″ at 3h. 57m. 52s. p.m.; and second observation, sun's L. L. 31° 9′ 59″ at 4h. 7m. 20s. p.m. If the index error be − 1′ 25″, and height of the eye 20 feet, find the south latitude.

Ex. 490. 1887, September 3rd, in longitude 175° 27′ W., the means of observed altitudes and apparent times were at first observation, sun's L. L. 40° 24′ 54″ at 1h. 27m. 32s.; second observation, sun's L. L. 39° 8′ 57″ at 1h. 40m. Index error − 2′ 24″, height of the eye 19 feet. Required the south latitude.

Ex. 491. Investigate by means of figures the following rules:—

(*a*) To find latitude by sun's meridian altitude.

(*b*) To find latitude by double altitude.

(*c*) To find longitude by chronometer and altitude of star.

Royal Naval College, 1863.

Ex. 492. The position of the ship at sea is defined by the intersection of two determinate lines on the earth's surface, on both of which the ship is ascertained to be. Describe fully the different systems of these curves. Give a full explanation of Sumner's method, and show how soundings, sighting lights, &c., can be combined with a circle of equal altitudes to determine the position of the ship. *H.* 1869.

Ex. 493. Explain and prove the method of finding the latitude by double altitudes when the objects are two stars observed at different times, showing especially how the polar angle may be accurately determined.

Find the exact polar angle in the case of an observation of two stars x and y, the R. A. of x being 6h. 38m. 25s., and that of y being 13h. 17m. 11s., the altitude of y being taken 2h. 5m. 10s. after that of x as measured by chronometer.

H. 1878.

Ex. 494. Explain (by means of a figure) the rule for finding the latitudes from altitudes of two heavenly bodies not observed at the same time. What are the corrections that ought to be applied to the interval between the two observations?

Royal Naval College, 1864.

Ex. 495. Investigate a method for determining the latitude and longitude by means of two altitudes of heavenly bodies, and the times of observations. *H.* 1869.

Ex. 496. Explain the method of finding latitude by Sumner's method. *For Beaufort Testimonial*, 1864.

Ex. 497. Explain with figure, the method of finding latitude by double altitudes of two stars.

	Alt.	*Dec.*	*R. A.*	*Bearing.*
x	65°	55° N.	2hrs.	N.W.
y	50°	60° N.	7hrs.	N.E. by N.

The altitude of y was taken 25m. after that of x. Draw the figure neatly on the plane of the equator.

Royal Naval College, 1872.

Ex. 498. Investigate the correction to be applied to the latitude, as found from a double altitude of the sun by Ivory's method, in consequence of the change of declination in the interval between the observations.

At what times would Ivory's method, independently of corrections, give the best results, and at what times would the corrections be the greatest? *H.* 1882.

Ex. 499. June 17th, 1887, about 4h. 30m. p.m. in latitude by account 44° 10′ N., longitude 13° 30′ W., the sun's true altitude was found to be 31° 44′ 45″, the Greenwich mean time was June 17th, 5h. 25m. 21s. Required the longitude of two points on the curve of equal altitudes corresponding to the latitudes 43° 50′ N. and 44° 30′ N.

Show whether this curve of equal altitudes is closed or not, and find the distance of its centre from the equator.

H. 1875.

Ex. 500. Two known stars a, b differed 5m. in right ascension. The altitude of a was observed 5 minutes after that of b. Investigate a formula for obtaining the latitude.

Royal Naval College, 1872.

Ex. 501. In the problem for determining the latitude by altitudes taken within a few minutes of each other, investigate the formula for finding the angle of position.

Royal Naval College, 1867.

Ex. 502. Investigate a formula for obtaining the latitude by means of two altitudes taken within a few minutes of each other; when is this method useful, and under what conditions?

Royal Naval College, 1867.

Ex. 503. Explain briefly the method of finding the place of a ship known as Sumner's method. Two known stars, x and y, differ 10 minutes in R. A., that of y being the greater. The altitude of x was observed, and 10 minutes afterwards that of y. Show how the latitude may be determined.

For Lieutenant, 1874.

CHAPTER XVII.

Longitude by lunar distances—Why lunar distances are so useful—How Greenwich mean time is found from lunar distances—Proportional logarithms—Proof—Which object should be chosen—Examples—Exercise—Clearing the distance—Two principles employed—Borda's method—Proof of formulæ—Rule—Example—Method of shortening work by rejecting the seconds—Krafft's method—Proof of formulæ—Rule—Example—Merrifield's method—Proof of formulæ—Rule—Example—Airy's method—Proof of formulæ—Example—Rigid method not used at sea—Example by all the methods—Precautions to be taken—Checks to work—Example by all the methods—Exercise—Examination.

By a lunar observation is meant the deducing of the time at Greenwich by measuring the distance between the moon and some other object which lies in or near her path. The chronometer, although brought to such a high state of efficiency, like every other machine made by man, is not a perfect instrument, but is liable to accidents and variations; and hence, though every precaution may be taken by those who have the care of it, its rate may alter, it may stop in winding, its spring may snap, or other injury may unavoidably befall it. We have already shown how its rate may be detected and allowed for; but when an injury occurs it is necessary to be able to substitute other means for finding the position of the ship, and it is expedient these means should be removed as far as possible from human control. Such can only be found in the "*mechanism of the heavens.*"

The moon's motion, as viewed from the earth, is quicker than that of any other celestial body. She completes a revolution from west to east among the fixed stars in 27d. 7h. 43m. 11·416s., which is at the average rate of about 33″ of space in every minute of time, or more than 13 times as fast as the sun; and, sometimes, when nearest the earth, her rate is more than 36″ per minute. This quick motion towards or from other objects suggests a means of measuring an interval of time; and, in fact,

the astronomer regards the heavens as the face of a clock, on which the sun, planets, and stars, near the moon's path, take the place of the figures, and the moon that of the hand moving about among them pointing out the time. Now, as it is the *rate* with which the moon approaches and recedes from other objects that makes it so peculiarly useful in determining time, it is obvious that if any other body in the heavens had a quicker motion it would be preferred to the moon ; and, did our satellite revolve as quickly as Jupiter's first one does, longitude with us would be deduced with equal precision as latitude. As distances can generally be measured within 30″ by competent observers, it follows that even now mean time at Greenwich can be determined to within one minute of time, which corresponds to 15′ of longitude.

Of late years the *lunar theory* has received so much attention from the best mathematicians and astronomers of all countries that now we may consider it practically perfect, as is evidenced by the publication of Hansen's tables. The position of the moon, like that of the sun and stars, may by its aid be calculated with the greatest precision. This is done from three to four years in advance, and the moon's distances from certain bodies, as seen from the centre of the earth, uninfluenced by refraction, are inserted on pages xiii. to xviii. of each month in the "Nautical Almanac." The bodies selected are the Sun, the four planets, Venus, Mars, Jupiter, and Saturn (Mercury is omitted because of its proximity to the sun and the outer planets, because they are invisible without telescopic aid), and the fixed stars, α Arietes, Aldebaran, Pollux, Regulus, Spica, Antares, α Aquilæ (Altair), Fomalhaut, and α Pegasi (Markab), all of which lie near the moon's path. The Sun, Jupiter, and Venus are especially serviceable for lunar observations, because from their brilliancy the altitudes of the two latter can be taken in pretty strong twilight when the horizon is perfectly visible, and hence at times when stars cannot be seen. From the difficulty in taking correct altitudes of the latter, it is better to compute them when they are to be used at sea for lunars. The distances, as applicable for observation, are registered for every three hours of Greenwich mean time, and the moon's motion is sufficiently uniform during these intervals to give an approximate Greenwich mean time by simple proportion when any other distance is known. But owing to the want of strict uniformity in the motion, a correction called that for "second

differences," p. 476, "Nautical Almanac," must be applied before the *correct* Greenwich mean time is fully determined. It will be shown hereafter that from an accurately measured distance on the surface of the earth how the true distance at the centre can be deduced; and then, by comparing the computed true distance with those given in the "Nautical Almanac," the great problem of finding, at any station, what the Greenwich time is, is solved without the aid of a chronometer or other instrument than the sextant. Lunars therefore become of the utmost importance to mariners, because of the frequent opportunities which they present for practising them, as they can always be used except when the moon is too close to the sun to be visible; and as the lunar is the only method for determining longitude by means of sextant observations independent of time-keepers, it is the most important and useful one, especially at sea.

Having calculated the true or geocentric distance between the moon and another suitable object, the Greenwich mean time is thus found:—

RULE (*a*) For the given date take from the "Nautical Almanac" the distances between which the calculated distance is found, and find their difference. This is the change in distance for three hours.

(*b*) Find the difference in the distance between that at the earlier hour in (*a*) and the calculated true distance. This is the change in distance for the required interval.

(*c*) Then form a simple proportion from having a change in distance from (*a*) produced in three hours, how long will it take to produce the change in (*b*)?

(*d*) Having found this, it must be added to the earlier hour used in finding the changes above.

This calculation may, as stated, be performed by simple arithmetic; but to facilitate the computation Dr. Maskelyne proposed the use of proportional logarithms, and he inserted them in the "Nautical Almanac" for that purpose.

PROPORTIONAL LOGARITHMS.—The proportional logarithm of a given quantity is the logarithm of the ratio of some constant greater than that quantity to the given quantity. Thus, if N be the quantity whose proportional logarithm is required to the constant A, A must be greater than N, and

$$\text{Prop. log. } N = \text{log. } \frac{A}{N}$$
$$= \text{log. } A - \text{log. } N.$$

The use of these logarithms in Nautical Astronomy is especially adapted to interpolating for the time at which a given lunar distance occurs; and as these distances are given in the "Nautical Almanac" for every three hours, the constant A in these tables equals three hours; then

$$\text{Prop. log. } N = \text{log. 3h.} - \text{log. } N \text{ in hours}$$
$$= \text{log. 10800s.} - \text{log. } N \text{ in seconds.}$$

Ex. 504. Find the proportional logarithm for 1h. 16m. 25s.

$$\text{Prop. log. 1h. 16m. 25s.} = \text{log. } \frac{\text{3h.}}{\text{1h. 16m. 25s.}}$$
$$= \text{log. } \frac{10800}{4585}$$
$$= \text{log. } 10800 - \text{log. } 4585$$
$$= 4{\cdot}033424 - 3{\cdot}661339$$
$$= {\cdot}372085. \quad \textit{Answer.}$$

The logarithms are registered only to four figures, because further than this conduces to no greater accuracy in practice:—

$$\therefore \text{ Prop. log. 1h. 16m. 25s.} = 3721.$$

In deducing the time corresponding to a true lunar distance by simple proportion, four entries to logarithmic tables are necessary for finding the interval elapsed since a tabulated distance in the "Nautical Almanac;" but by using proportional logarithms two entries will suffice, because the proportional logarithm of the change in distance for three hours is given opposite every distance in the "Nautical Almanac." The demonstration is as follows:—

Let d_1 and d_2 be the distances in the "Nautical Almanac" between which the true distance d lies, and t be the number of hours elapsed since the time for which d_1 is tabulated; then

$d_2 - d_1$ is the change in distance in three hours;
$d - d_1$ „ „ „ the required interval t;

$$\therefore d_2 - d_1 : d - d_1 :: 3 : t,$$
$$\text{or } \frac{3}{t} = \frac{d_2 - d_1}{d - d_1}$$
$$= \frac{3}{d - d_1} \div \frac{3}{d_2 - d_1}.$$

Hence by definition

Prop. log. t = prop. log. $(d - d_1)$ — prop. log. $(d_2 - d_1)$, the last of which quantities, viz. prop. log. $(d_2 - d_1)$, is tabulated in the "Nautical Almanac" opposite the distance d_1.

The method of finding Greenwich mean time from a lunar distance is explained in the "Nautical Almanac" at p. 499 in these words :—

Lunar distances. Pages xiii. *to* xviii. *of each month.*—These pages contain, for every third hour of Greenwich mean time, the angular distances, available for the determination of the longitude, of the apparent *centre* of the moon from the sun, the larger planets and certain stars as they would appear from the centre of the earth. When a lunar distance has been observed, and reduced to the centre of the earth, by clearing it of the effects of parallax and refraction, the numbers in these pages enable us to ascertain the exact Greenwich mean time at which the objects would have the same distance. They are arranged from *west* to *east*, commencing each day with the object which is at the greatest distance *westward* of the moon, in the order in which they appear: W. indicating that the object is west, and E. east of the moon.

The columns headed "P. L. of diff." contain the proportional logarithms of the differences of the distances at intervals of three hours, which are used in finding the Greenwich time corresponding to a given distance, according to the following rule, viz. :—Take the difference between the reduced distance and the *nearest* distance *preceding it*, in order of time, in the ephemeris; from the proportional logarithm of this difference subtract the proportional logarithm in the ephemeris; the remainder will be the proportional logarithm of a portion of time to be added to the hour answering to the *nearest* preceding distance, to obtain the approximate Greenwich mean time corresponding to the given distance.

If the distance between the moon and a star increased or decreased uniformly, the Greenwich times corresponding to a given distance as found by the above rule would be correct; but, as this is not the case, a correction must be applied to the time so found for the variation of the differences of the distances. This correction may be obtained by means of the table at page 476 in the following manner:—

(1) Find the approximate interval by the preceding rule.

(2) Take the differences between the proportional logarithm

following the distance in the ephemeris, and the proportional logarithms immediately to the left and right, and note the *mean* of these differences.

(3) With the approximate interval and this *mean* difference, as arguments, take out the correction from the table.

(4) If the proportional logarithms are *decreasing*, *add* the correction to the approximate time; but if *increasing*, *subtract* it: the result will be the Greenwich mean time.

Example I.—Suppose it were required to find the Greenwich mean astronomical time at which the *reduced* distance between the moon and Fomalhaut would be 47° 12′ 8″, about July 12th, 1887. It appears, by inspecting the distances, that the *nearest* distance *preceding* it, in order of time, is that on July 12th, at *midnight*; therefore,

Distance at midnight	46° 32′ 42″	and P. L. . .	3586
Reduced distance	47 12 8		
Difference	0 39 26	. . . . P. L. . .	6594
Approximate interval	1h. 30m. 3s.	. . . P. L. . .	3008

The proportional logarithm following the distance in the ephemeris is 3586, those immediately to the left and right are respectively 3614 and 3558; the differences are therefore 28 and 28, and the *mean* of them 28.

Opposite to 1h. 30m. 3s. (or the quantity nearest to it, 1h. 30m.), and under 28 in the table, we have for the correction 9s., which, *added* to the approximate interval 1h. 30m. 3s., because the proportional logarithms are *decreasing*, gives 1h. 30m. 12s. for the true interval after *midnight;* the Greenwich mean astronomical time is therefore July 12th, 13h. 30m. 12s. The omission of this correction would produce an error of 2′ in the longitude.

It will sometimes happen that the *mean* difference of the proportional logarithms will exceed the limit of the table of correction: in this case the table may be entered with the approximate interval, and *any fraction* of the difference and the corresponding correction *increased in like proportion.*

Example II.—Suppose it were required to find the Greenwich mean astronomical time at which the *reduced* distance between the moon and Fomalhaut would be 28° 6′ 4″, about September

3rd, 1887. By inspecting the distances it appears that the nearest preceding is on September 3rd, at III.; therefore,

Distance at III.	27° 29′ 35″ and P. L. . .	4634
Reduced distance	28 6 4	
Difference	0 36 29 . . . . P. L. . .	6932
Approximate interval	1h. 46m. 2s. . . . P. L. . .	2298

The *mean* difference between the proportional logarithms determixed as in the preceding example is 155, one-half of which is 78: under 78 in the table, and opposite 1h. 50m., is 23s.: the correction is therefore 46s. to be *added* to the approximate interval, because the proportional logarithms are *decreasing;* the Greenwich mean astronomical time is therefore September 3rd, 4h. 46m. 48s. The omission of the correction would produce an error of 11·5′ in the longitude: it may, however, be considered as an extreme case, and such as will seldom be met with.

The proportional logarithms also indicate the star most favourably circumstanced for observation, that star being preferred which has the least proportional logarithm opposite to it; for the greater the velocity of the moon from or towards a star, the greater is the reliance to be placed on an observation of the distance; and as proportional logarithms decrease as their natural numbers increase, a smaller proportional logarithm indicates a greater velocity of the moon, or a greater variation of distance in the interval, upon which the value of the observation depends. It is not to be inferred from these remarks that observations of any of the distances are to be neglected; on the contrary, every registered star should invariably be observed when possible.

Another consideration which should guide an observer in the choice of an object for a lunar observation, is whether the object has nearly the same declination as the moon; and those should receive the preference the difference between whose declination and that of the moon is as small as possible when compared with the difference in their right ascensions.

In the preceding explanation taken from the "Nautical Almanac" the use of *second differences* is seen from the errors which would ensue if these are neglected. The formation of the

table at page 476 in the "Nautical Almanac" may be gleaned from the following explanation.

SECOND DIFFERENCES.—Second differences are those arising from taking the difference between the successive differences of any varying function: thus, if a, b, c, d, e, &c., be the successive values of any varying function, then we get

Values of functions.	First differences.	Second differences.	Third differences.	Fourth difference.
a				
	$a-b$			
b		$a-2b+c$		
	$b-c$		$a-3b+3c-d$	
c		$b-2c+d$		$a-4b+6c-4d+e$
	$c-d$		$b-3c+3d-e$	
d		$c-2d+e$		
	$d-e$			
e				

By examining each successive difference above a law is easily seen; viz. the coefficient of each difference is formed in the same manner as the coefficients in expanding any binomial expression whose terms are separated by a $-$ sign: so that

$$n\text{th difference} = a - nb + \frac{n(n-1)}{1\,.\,2}c - \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}d, \text{ \&c.}$$

It will be found in actual practice that if the differences be carried far enough they all tend to equality; and when that is attained the value of the function can be ascertained for any given intermediate quantity. The method of differences is especially valuable in interpolating between the elements taken from the "Nautical Almanac," where the time is the independent variable; because the rates of variation of the sun's and moon's motions in right ascension and declination, the moon's rate of motion towards or from objects in her path, &c., all vary; and, therefore, the exact position of these objects at any intermediate time to those given in the "Nautical Almanac" cannot be determined by interpolating by simple proportion; but it will be found by carrying the interpolation to the second differences that a result is obtained sufficiently exact for all practical purposes. The second difference is, therefore, the rate at which the rate of varying varies.

In *Boole's finite differences*, Chapter III., the method of interpolation is fully explained for n differences. The formula applied to interpolating the time for lunar distances is

$$d = a + \Delta_1 t + \Delta_2 \frac{t(t-1)}{1 \cdot 2} + \&c.,$$

where d is the true lunar distance, a the distance from the "Nautical Almanac" at the hour preceding the required distance, Δ_1 the first difference, Δ_2 the second difference, and t the required interval in three hour units corresponding to the difference between the true distance d and distance at the preceding hour a.

Ex. 505. We will apply this method to the solution of the question given in the "Nautical Almanac," page 501 :—

Time.	Distances.	First differences.	Second difference.
III.	27° 29′ 35″		
		+ 1° 1′ 56″	
VI.	28 31 31		+ 2′ 6″
		+ 1° 4 2	
IX.	29 35 33		

The general formula is :—

$$d = a + \Delta_1 t + \Delta_2 \frac{t(t-1)}{1 \cdot 2},$$

where
$$\begin{aligned} d &= 28° \;\; 6' \;\; 4'' \\ a &= 27 \;\; 29 \;\; 35 \\ \Delta_1 &= \;\; 1 \;\;\; 1 \;\; 56 = 3716'' \\ \Delta_2 &= \qquad\; 2 \;\;\; 6 = \;\; 126. \end{aligned}$$

Then

$$\begin{aligned} 28° \, 6' \, 4'' &= 27° \, 29' \, 35'' + 3716'' t + 126'' \frac{t(t-1)}{1 \cdot 2} \\ 2189 &= 3716\, t + 63\, t^2 - 63\, t \\ &= 3653\, t + 63\, t^2 ; \\ \therefore t &= \frac{2189}{3653} - \frac{63}{3653} t^2. \end{aligned}$$

Hence first approximation $t = \dfrac{2189}{3653}$;

second ,, $t = \dfrac{2189}{3653} - \dfrac{63}{3653}\left(\dfrac{2189}{3653}\right)^2$
$= \cdot 59304059$ of 3 hours
$=$ 1h. 46m. 44·84s.

The true interval given in the "Nautical Almanac" is 1h. 46m. 48s., with which the method illustrated above would agree if a true value of t had been obtained from the third differences instead of the approximation used.

Ex. 506. To further illustrate the method of interpolation by second differences we will apply it to find the true declination of the sun at 8h. apparent time at Greenwich, on August 24th, 1887, having given the declination on the noons of the 23rd, 24th, 25th, and 26th of the month :—

Date.	Declination.	First differences.	Second differences.
23rd Aug.	11° 28′ 38·5″ N.		
		− 20′ 26·6″	
24 "	11 8 11·9 N.		+ 10·5″
		− 20 37·1	
25 "	10 47 34·8 N.		+ 10·2
		− 20 47·3	
26 "	10 26 47·5 N.		

3)61′ 51″ — 20·37 — 2)20·7″ — 10·35 second diff.

Mean of first differences − 1237″

$$\therefore \text{ dec.} = 11° \, 8' \, 11{\cdot}9'' - 1237\,t + 10{\cdot}35\,\frac{t\,(t-1)}{1\,.\,2}$$

where $t = \frac{1}{3}$ of a day.

$$\text{Hence dec.} = 11° \, 8' \, 11{\cdot}9'' - 1237 \times \tfrac{1}{3} + \frac{10{\cdot}35}{2} \times \tfrac{1}{3} \times \tfrac{2}{3}$$

$$= 11 \;\; 8 \;\; 11{\cdot}9 \; - 412{\cdot}33'' + 1{\cdot}15''$$

$$= 11 \;\; 1 \;\; 19{\cdot}7 \text{ N.}$$

which agrees with the declination when corrected by the hourly differences within one-fiftieth of a second.

EXERCISE XXIII.

Ex. 507. What Greenwich mean astronomical time corresponds to a reduced distance of 118° 49′ 52″ between the sun and moon about January 13th, 1887 ?

Ex. 508. Find the Greenwich mean time on May 24th, 1887, when the reduced distance in a sun lunar was 19° 33′ 41″.

Ex. 509. 1887, November 29. Find the Greenwich mean

astronomical time corresponding to a reduced distance of 28° 53′ 17″ between the moon and α Arietes.

Ex. 510. The reduced distance between the moon and α Pegasi was 31° 39′ 51″ on August 4th, 1887. Find the Greenwich mean astronomical time.

Ex. 511. Prove the above four examples by working by "second differences" as in Ex. 505.

Ex. 512. The moon's right ascension on January 23rd, 1887, was as follows:—

At 4h.	19h.	59m.	4·33s.
„ 5	20	1	13·27
„ 7	20	5	30·65
„ 8	20	7	39·08

Find her R. A. by means of second differences on January 23rd, at 6 hours.

Ex. 513. Find the proportional logarithms for 30m. 27s., for 1h. 53m. 42s., and for 2h. 47m. 38s.

Ex. 514. Why are the proportional logarithms of the difference of the distances of the moon from a star not the same for two consecutive three hourly intervals?

Royal Naval College, 1872.

Ex. 515. How can you tell by looking at the proportional logarithms in the monthly table of "lunar distances" in the "Nautical Almanac," which is the star most favourably circumstanced for accurate observations?

Royal Naval College, 1872.

Ex. 516. The lunar distances of certain stars being given in the "Nautical Almanac" for every third hour, show the use of proportional logarithms in finding the time corresponding to some intermediate distance. *Royal Naval College*, 1868.

Ex. 517. Describe fully the table in the "Nautical Almanac," pp. xiii.—xviii., "lunar distances," and explain its use. What heavenly bodies are chosen for this table, and why? How is it that the moon alters her distance more rapidly for some of these bodies than for others? *A*. 1879.

Ex. 518. How is it that the right ascension and declination have to be given in the "Nautical Almanac" for the moon at intervals so much smaller than for the sun? Explain clearly the cause of this. What is the largest amount of the sun's declination, and why? What bodies are chosen for lunar distances, and why?

Find the distance of the moon from Fomalhaut on April 11th, 1887, at 4h. 57m. 25s. G. M. T. Do this (1) without the use of logarithm tables, (2) by proportional logarithms.

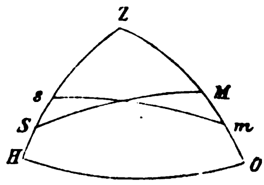
H. 1880.

CLEARING THE DISTANCE.

The altitudes of all objects, except the fixed stars, are affected by the two physical causes *refraction* and *parallax*, of which the former only produces an effect on the altitudes of stars. Hence to reduce an altitude taken at the surface to what it would be uninfluenced by refraction at the centre of the earth, the effects of these two causes must be calculated and allowed for. This is done thus:—

Let Z be the zenith of the observer, and $H\ O$ his horizon, m the apparent place of the moon, and s the apparent place of the sun, planet, or star. Then because refraction and parallax always produce their effects in vertical circles, the correction of altitude will always be the difference between these. Now, in the case of the moon, the parallax in altitude is always greater than refraction, hence the elevating effect of the former is greater than the depressing effect of the latter, and therefore the true place of the moon is above her apparent place. With all other celestial objects the effect of refraction is greater than that of parallax, and therefore the true place of the sun, planet, or star is below that of its apparent place. Make $m\ M$ equal to the difference between the apparent and true places of the moon: also make $s\ S$ the difference between the apparent and true places of the other object. Join $m\ s$ and $M\ S$ by arcs of great circles; then $m\ s$ is the apparent distance, or that which is seen between the centres of the objects at the surface of the earth, and $M\ S$ is the true or geocentric distance, or that at the centre of the earth uninfluenced by refraction. Reducing the apparent to the true distance, or calculating the effects of refraction and parallax on the apparent distance, is technically called *clearing the distance*. Various methods have been proposed by different persons, and Woodhouse says in his Astronomy that Delambre has given twenty different formulæ for that purpose; but all the methods in use reduce themselves in principle

Fig 57.



to two: and the rules given by different authors are obtained by using different trigonometrical transformations.

I. *First principle.*—From the two apparent zenith distances $Z\,m$ and $Z\,s$, and the apparent distance $m\,s$, all of which are immediately deduced from observations made on the surface of the earth, the zenithal angle Z is calculated. Then with the true zenith distances $Z\,M$ and $Z\,S$ obtained from the apparent zenith distances, together with the angle Z, the true distance $M\,S$ is easily found.

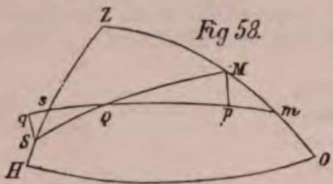


Fig 58.

II. *Second principle.*—Let $m\,M$ and $s\,S$ be the corrections of the altitudes as before; then in the triangle $Z\,m\,s$ the three sides are known, and the angles at m and s can easily be found by the usual formulæ. From the point Q, where the true and the apparent distances intersect, as pole, and with radii $Q\,M$ and $Q\,S$ describe arcs of small circles $M\,p$ and $S\,q$, then $m\,p$ will be the reduction of the apparent distance due to the moon's correction of altitude, and $s\,q$ is the reduction due to the correction of altitude of the other object. In the triangles $M\,m\,p$, $S\,s\,q$ right angled at p and q, the angles at m and s are known as well as $M\,m$ and $S\,s$, the corrections for altitude; and as these triangles are always very small, $M\,m$ never amounting to sixty minutes, they may be considered as plane; then

$m\,p$ = correction of moon's alt. × cos. m,

$s\,q$ = correction of sun, planet, or star's alt. × cos. s.

From the figure it is evident, when m and s are acute, $m\,p$ is subtractive and $s\,q$ additive, and contrary when these are obtuse. A modification of this principle is sometimes made by calculating the effects of refraction and parallax separately on the apparent distance for each object.

Because of the tediousness of the operations involved in clearing the apparent distance from the effects of refraction and parallax by the direct formulæ from spherical trigonometry, many eminent astronomers and mathematicians have deduced rules for facilitating the solution of the problem, among whom the best known are Airy, Borda, La Caille, Delambre, Dunthorne, Krafft, Legendre, Lelande, Lyons, Maskelyne, Mendoza Rios, and Witchell. The methods in general use depending

on the *first principle* are those by Jean Charles Borda, published in his treatise, "On the Reflecting Circle," in 1787, and that by Krafft, of St. Petersburg, in 1791. The former is considered by competent authorities as the best, and Mr. Riddle says of it: "Borda's method is the most simple and direct, and gives the corrected distance without embarrassment of algebraical signs; and, moreover, has the advantage of requiring no special tables in its application." But to those who possess a table of *natural versed sines*, Krafft's method of clearing the distance is exceedingly simple, as no logarithmic tables are required, and the true distance is deduced from versed sines only.

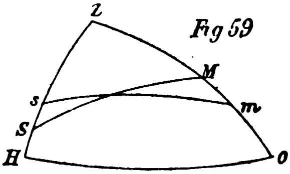


Fig 59

The investigation of Borda's method is as follows:—

In the triangle $Z\,S\,M$—

$$\cos. Z = \frac{\cos. SM - \cos. ZS . \cos. ZM}{\sin. ZS . \sin. ZM}$$

$$= \frac{\cos. SM - \sin. HS . \sin. OM}{\cos. HS . \cos. OM} \quad . \text{ I.}$$

In triangle $Z s m$: $$\cos. Z = \frac{\cos. s m - \cos. Z s . \cos. Z m}{\sin. Z s . \sin. Z m}$$

$$= \frac{\cos. s m - \sin. H s . \sin. O m}{\cos. H s . \cos. O m} \quad \text{II.}$$

Let m and m' be the true and apparent altitudes of the moon.
s and s' „ „ „ sun.
d and d' „ „ distances.

Equating I. and II.—

$$\frac{\cos. d - \sin. s . \sin. m}{\cos. s . \cos. m} = \frac{\cos. d' - \sin. s' . \sin. m'}{\cos. s' . \cos. m}.$$

Adding unity to each side—

$$\frac{\cos. d + \cos. (m + s)}{\cos. s . \cos. m} = \frac{\cos. d' + \cos. (m' + s')}{\cos. s' . \cos. m'} \quad . \text{ III.}$$

$$\frac{\left(1 - 2 \sin.^2 \frac{d}{2}\right) + \left(2 \cos.^2 \frac{m+s}{2} - 1\right)}{\cos. s . \cos. m.} = \frac{2 \cos. \frac{m'+s'+d'}{2} . \cos. \frac{m'+s'-d'}{2}}{\cos. s' . \cos. m'.}$$

$$\therefore \sin.^2 \frac{d}{2} = \cos.^2 \frac{m+s}{2} - \frac{\cos. s . \cos. m}{\cos. s' . \cos. m'} . \cos. \frac{m'+s'+d'}{2} . \cos. \frac{m'+s'-d'}{2}$$

$$\text{Let } \cos.^2 \theta = \frac{\cos. s . \cos. m}{\cos. s' . \cos. m'} . \cos. \frac{m'+s'+d'}{2} . \cos. \frac{m'+s'-d'}{2} \quad . \text{ A.}$$

$$\text{Then } \sin.^2 \frac{d}{2} = \cos.^2 \frac{m+s}{2} - \cos.^2 \theta$$
$$= \tfrac{1}{2}\{1 + \cos.(m+s)\} - \tfrac{1}{2}\{1 + \cos. 2\theta\}$$
$$= \tfrac{1}{2}\{\cos.(m+s) - \cos. 2\theta\}$$
$$= \sin.\left(\frac{m+s}{2} + \theta\right). \sin.\left(\frac{m+s}{2} - \theta\right);$$
$$\therefore \sin. \frac{d}{2} = \sqrt{\sin.\left(\frac{m+s}{2} + \theta\right). \sin.\left(\frac{m+s}{2} - \theta\right)} \quad . \text{ B.}$$

Equations A and B are those used in the solution of the problem, and adapted to logarithmic computation are as follows :—

$$\log. \cos. \theta = \tfrac{1}{2}\{\log. \sec. m' + \log. \sec. s' + \log. \cos. \frac{m'+s'+d'}{2} + \log. \cos. \frac{m'+s'-d'}{2} + \log. \cos. m + \log. \cos. s\},$$

$$\text{and } \log. \sin. \frac{d}{2} = \tfrac{1}{2}\left\{\log. \sin.\left(\frac{m+s}{2} + \theta\right) + \log. \sin.\left(\frac{m+s}{2} - \theta\right)\right\}.$$

A modification of the formulæ is obtained by assuming

$$\sin.^2\theta = \frac{\cos. s \,.\, \cos. m}{\cos. s' \,.\, \cos. m'} . \cos. \frac{m'+s'+d'}{2} . \cos. \frac{m'+s'-d'}{2} \text{ in Equa. A.}$$

and Equation B will then take the form of

$$\sin. \frac{d}{2} = \sqrt{\cos.\left(\frac{m+s}{2} + \theta\right). \cos.\left(\frac{m+s}{2} - \theta\right)}.$$

On formulæ A and B the following rule depends :—

RULE (*a*) Find the true (m and s) and apparent (m' and s') altitudes of the moon and the sun, planet, or star, whichever may have been used.

(*b*) Correct the observed distance for index error and semidiameter.

(*c*) Place under one another the apparent distance d' and the two apparent altitudes m' and s', and take half their sum S; from the half sum subtract the apparent distance d' Under this place the true altitudes m and s.

(*d*) Take from the tables log. secant m', log. secant s', and log. cosines of S, $S - d'$, m and s. Add these six quantities together and divide by two, the result is log. cosine θ.

$$\log. \cos. \theta = \tfrac{1}{2}\{\log. \sec. m' + \log. \sec. s' + \log. \cos. \frac{m'+s'+d'}{2} + \log. \cos. \frac{m'+s'-d'}{2} + \log. \cos. m + \log. \cos. s\}.$$

(*e*) Take half the sum of the true altitudes m and s, call this ϕ; find the sum and difference of θ and ϕ, then add together sines of this sum and difference, divide by two; the result is the sine of half the true distance.

$$\text{log. sin. } \frac{d}{2} = \tfrac{1}{2}\left\{\text{log. sin. }\left(\frac{m+s}{2}+\theta\right) + \text{log. sin.}\left(\frac{m+s}{2}-\theta\right)\right\}.$$

Ex. 519. Given the undermentioned data to compute the true distance between the sun and moon.

Apparent altitudes.	*True altitudes.*	*Apparent distance.*
m' 13° 29′ 27″	m 14° 18′ 32″	d' 107° 52′ 4″
s' 31 11 34	s 31 10 7	

$$\left.\begin{aligned} \cos.\theta &= \sqrt{\sec.m'.\sec.s'.\cos.S.\cos.(S-d').\cos.m.\cos.s} \quad . \quad \text{A.} \\ \phi &= \frac{m+s}{2} \\ \sin.\frac{d}{2} &= \sqrt{\sin.(\theta+\phi).\sin.(\theta-\phi)} \quad . \; . \; . \; . \; . \; . \; . \quad \text{B.} \end{aligned}\right\}$$

d'	107°	52′	4″	
m'	13	29	27	sec. ·012151
s'	31	11	34	sec. ·067815
	2)152	33	5	
S	76	16	32·5	cos. 9·375207
$S \backsim d'$	31	35	31·5	cos. 9·930337
m	14	18	32	cos. 9·986314
s	31	10	7	cos. 9·932295
ϕ	22	44	19·5	2)19·304119
θ	63	19	58·3	cos. 9·652059
$\theta+\phi$	86	4	17·8	sin. 9·998978
$\theta-\phi$	40	35	38·8	sin. 9·813378
				2)19·812356
$\frac{d}{2}$	53	40	43·2	sin. 9·906178
			2	
d	107	21	26·4	True distance.

The trouble of taking out the logarithms for seconds in the first four quantities may be thus obviated. Add or subtract the same number of seconds to the apparent and true altitudes, so as to eliminate the seconds in the apparent altitudes. In the present example if we subtract 27″ from the moon's altitudes we have $m' = 13° 29' 0''$ and $m = 14° 18' 5''$; similarly by subtracting 34″ from the sun's altitudes we get $s' = 31° 11' 0''$ and $s = 31° 9' 33''$. Next add or subtract a number of seconds from the apparent distance so as to make $(d' + m' + s')$ an even number of minutes; in this instance reject 4″, which must be added again to the true altitude as found by the approximate data. The work will then stand thus:—

d'	107°	52′	0″		
m'	13	29	0	sec.	·012138
s'	31	11	0	sec.	·067772
	2)152	32	0		
S	76	16	0	cos.	9·375487
$S \backsim d'$	31	36	0	cos.	9·930300
m	14	18	5	cos.	9·986328
s	31	9	33	cos.	9·932338
ϕ	22	43	49		2)19·304363
θ	63	19	29	cos.	9·652181
$\theta + \phi$	86	3	18	sin.	9·998970
$\theta - \phi$	40	35	40	sin.	9·813381
					2)19·812351
$\frac{d}{2}$	53	40	41·6	sin.	9·906175
			2		
d	107	21	23·2		
Seconds rejected		+	4		
True dist.	107	21	27·2		

This agrees with the former calculation within eight-tenths of a second.

Krafft's method is thus deduced:—

Returning to Equation III., p. 297—

$$\cos. d = -\cos. (m+s) + \frac{\cos. m \,.\, \cos. s}{\cos. m' . \cos. s'} \{\cos. d' + \cos. (m'+s)\}.$$

Assume $\dfrac{\cos. m \,.\, \cos. s}{\cos. m' . \cos. s'} = 2 \cos. A$ . . . . . . (C)

Then

$$\begin{aligned} \cos. d &= -\cos. (m+s) + 2 \cos. A \{\cos. d' + \cos. (m'+s')\} \\ &= -\cos. (m+s) + 2 \cos. A . \cos. d' + 2 \cos. A . \cos. (m'+s') \\ &= -\cos. (m+s) + \cos. (d'+A) + \cos. (d' \backsim A) \\ &\qquad + \cos. \{(m'+s')+A\} + \cos. \{(m'+s') \backsim A\}; \end{aligned}$$

$$\begin{aligned} \therefore\ & 1 - \cos. d \\ &= \{1 + \cos. (m+s)\} + \{1 - \cos. (d'+A)\} + \{1 - \cos. (d' \backsim A)\} \\ &\quad + [1 - \cos. \{(m'+s')+A\}] + [1 - \cos. \{(m'+s') \backsim A\}] - 4, \end{aligned}$$

$$\begin{aligned} \text{and vers. } d &= \text{suvers. } (m+s) + \text{vers. } (d'+A) + \text{vers. } (d' \backsim A) \\ &\quad + \text{vers. } \{(m'+s')+A\} + \text{vers. } \{(m'+s') \backsim A\} - 4 \\ &= \text{vers. (sum true z. d.)} + \text{vers. } (d'+A) + \text{vers. } (d' \backsim A) \\ &\quad + \text{vers. } \{(m'+s')+A\} + \text{vers. } \{(m'+s') \backsim A\} - 4 . \text{ (D)}. \end{aligned}$$

Equations C and D give the full solution of the problem by Krafft's method.

The angle A in equation C is called by Inman the auxiliary angle A, and is computed so as to be taken out at once from his tables. Those who have not a copy of Inman's tables can easily calculate the value of A for themselves from the formulæ.

RULE (*a*) Obtain the apparent and true altitudes of the objects and the apparent distance between them as in Borda's method.

(*b*) Find the sum of the apparent altitudes and of the true zenith distances.

(*c*) From the sum of log. cos. m, log. sec. m', log cos. s, and log. sec. s' subtract log. 2; the remainder is log. cos. of the auxiliary angle A, which is always a little more than 60°.

(*d*) Add together vers. $(d' + A)$, vers. $(d' \backsim A)$, vers. $\{(m' + s') + A\}$, vers. $\{(m' + s') \backsim A\}$, and vers. (sum true zen. dist.), and subtract 4 from the sum; the remainder is versine of the true distance.

Ex. 520. Given the undermentioned data to compute the true distance between the sun and moon.

Apparent altitudes.	*True altitudes.*	*Apparent distance.*
m' 13° 29′ 27″	m 14° 18′ 32″	d' 107° 52′ 4″
s' 31 11 34	s 31 10 7	

The student will see this is the same question we have already worked.

$$Cos.\ A = \tfrac{1}{2}\ cos.\ m\ .\ sec.\ m'\ .\ cos.\ s\ .\ sec.\ s'.$$

m	14° 18′ 32″	cos.	9·986314
m'	13 29 27	sec.	·012151
s	31 10 7	cos.	9·932295
s'	31 11 34	sec.	·067815
			9·998575
		2 log.	·301030
A	= 60 6 30	cos.	9·697545
d'	= 107 52 4		
$s' + m'$	= 44 41 1		
$Z + z$	= 134 31 21		

$$Vers.\ d = vers.\ (sum\ true\ z.\ d.)\ .\ vers.\ (d' + A)\ vers.\ (d' \infty A).$$
$$vers.\ \{(m' + s') + A\}\ .\ vers.\ \{(m' + s') \infty A\}.$$

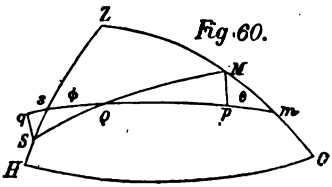
$Z + z$	= 134° 31′ 21″	vers.	1·701117 +	73
$(d' + A)$	= 167 58 34	vers.	1·978026 +	36
$(d' \infty A)$	= 47 45 34	vers.	·327633 +	121
$(m' + s') + A$	= 104 47 31	vers.	1·255165 +	145
$(m' + s') \infty A$	= 15 25 29	vers.	·035982 +	37
	Parts for seconds		412	412
True distance	107 21 27	vers.	5·298335	

By comparison it will be seen that the true distance agrees with that already obtained within a few tenths of a second.

The following solution of the problem depends on the second principle, and is believed to be new. Its practical application is as simple and direct as that of Borda's, as it requires no distinction of cases, "is without embarrassment of signs, and, moreover, has the advantage of requiring no special tables in its application." It was submitted by the author of the present treatise to the Royal Astronomical Society, and published in the *Monthly Notices* of that Society for April, 1884.

Let Z be the zenith of the observer, $Z\ O$ the vertical circle through the moon, and $Z\ H$ that through the other body. Let M be the true place, and m the apparent place of the moon; S the *true, and* s the apparent places of the other body. Join $S\ M$ and

$s\,m$, then $S\,M$ is the true distance and $s\,m$ the apparent distance between the objects. From Q, where $S\,M$ and $s\,m$ intersect, as pole; draw $M\,p$ and $S\,q$ portions of small circles: the angles at p and q are right angles.



Let d' be the app. distance.
m' „ moon's app. alt.
z „ „ „ z. d.
s' „ object's „ alt.
z' „ „ „ z. d.

$$S' = \frac{z + z' + d'}{2}. \qquad S = \frac{m' + s' + d'}{2}.$$

C = moon's correction for altitude = par. in alt. − refr.
c = object's „ „ „ = par. in alt. − refr.
θ = angle $Z\,m\,s$.
ϕ = angle $Z\,s\,m$.

Then true distance
$$\begin{aligned} S\,M = qp &= sm - pm + sq \\ &= d' - C\,.\cos.\,\theta + c\,.\cos.\,\phi \\ &= d' - C\left(1 - 2\sin.^2\frac{\theta}{2}\right) + c\left(1 - 2\sin.^2\frac{\phi}{2}\right); \end{aligned}$$

$$\therefore\ d = d' - (C - c) + 2\left(C.\sin.^2\frac{\theta}{2} - c\,.\sin.^2\frac{\phi}{2}\right) \quad \text{I.}$$

$$\begin{aligned} \text{Now } \sin.^2\frac{\theta}{2} &= \frac{\sin.\,(S' - d')\,.\,\sin.\,(S' - z)}{\sin.\,d'\,.\,\sin.\,z} \\ &= \frac{\sin.\,\dfrac{z' + z - d'}{2}\,.\,\sin.\,\dfrac{z' + d' - z}{2}}{\sin.\,d'\,.\,\cos.\,m'} \\ &= \frac{\sin.\left(90 - \dfrac{m' + s' + d'}{2}\right).\,\sin.\,\dfrac{m' + d' - s'}{2}}{\sin.\,d'\,.\,\cos.\,m'} \\ &= \frac{\cos.\,S\,.\sin.\,(S - s')}{\sin.\,d'\,.\,\cos.\,m'}. \end{aligned}$$

$$\text{Similarly, } \sin.^2\frac{\phi}{2} = \frac{\cos.\,S\,.\,\sin.\,(S - m')}{\sin.\,d'\,.\,\cos.\,s'};$$

$$\begin{aligned} \therefore\ C\,.\sin.^2\frac{\theta}{2} - c\,.\sin.^2\frac{\phi}{2} &= \frac{\cos.\,S}{\sin.\,d'}\,.\left(\frac{C\,.\sin.\,(S - s')}{\cos.\,m'} - \frac{c\,.\sin.\,(S - m')}{\cos.\,s'}\right) \\ &= \text{cosec.}\,d'\,.\,\cos.\,S\,(M - N), \end{aligned}$$

where $\underline{M = C\,.\,\sec.\,m'\,.\,\sin.\,(S - s')}$,
and $\underline{N = c\,.\,\sec.\,s'\,.\,\sin.\,(S - m')}$.

Then from I.—

True distance $= d' - (C - c) + 2 \text{ cosec. } d' . \cos. S (M - N)$.

This formula may be modified by substituting $2 \cos.^2 \frac{\theta}{2} - 1$ for $\cos. \theta$, and $2 \cos.^2 \frac{\phi}{2} - 1$ for $\cos. \phi$ in the beginning.

RULE (*a*) Place the moon's apparent altitude, the other object's apparent altitude, and the apparent distance, under one another, and take half their sum (S); from which subtract the sun's apparent altitude ($S - s'$) and the moon's apparent altitude ($S - m'$). Under these place the moon's correction for altitude (C) and the correction for the altitude of the other object (c).

(*b*) Add together logs. sec. moon's apparent altitude, sin. ($S - s'$), and moon's correction reduced to seconds; the sum is log. the number of seconds in M.

$$M = C . \sec. m' . \sin. (S - s').$$

(*c*) Add together logs. sec. other object's apparent altitude, sin. ($S - m'$), and other object's correction reduced to seconds; the sum is log. of the number of seconds in N.

$$N = c . \sec. s' . \sin. (S - m').$$

(*d*) Find the difference between M and N, and add together logs. cosec. apparent distance, cos. S and $M - N$; the sum is log. $C. \sin.^2 \frac{\theta}{2} - c . \sin.^2 \frac{\phi}{2}$. in seconds.

(*e*) Double the result in d, and add to the apparent distance (d'), from which subtract the difference of correction for altitudes; the remainder is the true distance.

Ex. 521. Given the undermentioned data to compute the true distance between the sun and moon.

Apparent altitudes.	*True altitudes.*	*Apparent distance.*
m' 13° 29' 27"	m 14° 18' 32"	d' 107° 52' 4"
s' 31 11 34	s 31 10 7	

This is the same question as we have already used to illustrate the other two methods.

Here $C =$ correction for moon's alt. $= m - m' = 49'\ 5''$
$= 2945''$,
and $c =$ correction for sun's alt. $= s' - s = 1\ 27$
$= 87''$.

$$\textit{True distance} = d' - (C - c) + 2\left(C \,.\, \sin^2\frac{\theta}{2} - c \,.\, \sin^2\frac{\phi}{2}\right).$$

m'	13° 29′ 27″	sec. ·012151		
s'	31 11 34		sec. ·067815	
d'	107 52 4			cosec. ·021470
	2)152 33 5			
S	76 16 32·5			cos. 9·375207
$S - s'$	45 4 58·5	sin. 9·850113		
$S - m'$	62 47 5·5		sin. 9·949046	
C	49′5″=2945″	log. 3·469085		
c	1 27 = 87		log. 1·939519	
M	2144·6 =	log. 3·331349		
		N 90·4	log. 1·956380	
			$M - N$ 2054·2″	log. 3·312642

$C \,.\, \sin^2\frac{\theta}{2} - c \,.\, \sin^2\frac{\phi}{2}$	512·1	log. 2·709319
	= 8′ 32·1″	
	2	
	+ 17 4·2	
$C - c$	− 47 38	
Corr. for app. dist.	− 30 33·8	
d'	107 52 4	
True distance d	107 21 30·2	

This again is seen to agree with the other results within 3 seconds.

In this method the work may be shortened in a precisely similar manner to that by Borda's method, viz. by neglecting or adding seconds so as to make the altitudes, distance, and half sum an even number of minutes. It will then be found

that no logarithm need be taken out for seconds. In finding the true distance the same precaution must then be taken as before with the seconds in the distance by taking in or rejecting those before rejected or added.

The last method we shall give a demonstration of is the one by the late Astronomer-Royal, Sir G. B. Airy, K.C.B., F.R.S., by whose kind permission it is here inserted in full as published in a pamphlet by the Hydrographic Office at the Admiralty. * He says:—

I offer for the consideration of nautical men a method of correction of lunar distances which appears to possess the following properties. It is direct and simple in application. Using 5-figure-logarithms, and with the most ordinary care in the assumption of one element, it gives the result with an accuracy which is sensibly perfect. If the comparison of the result with the assumed element shows that the latter was erroneous, a repetition of the operation, with very few changes of figures, will make the result accurate.

The characteristic circumstance upon which this treatment depends is the use, in the factors of corrections, not of each apparent element nor of the corresponding correcting element, but of the mean between the two.

The elements which we require are—the apparent altitude and the corrected altitude of the moon, the apparent altitude and the corrected altitude of the sun, and the apparent and corrected distance. The first five of these are known accurately. The last (the corrected distance between the sun and the moon) must be estimated. There is no difficulty in doing this, with accuracy abundantly sufficient for this investigation. With Greenwich time by account, the distance may be rudely computed from the distances in the "Nautical Almanac." Or, without time or calculation, a navigator accustomed to lunar distances may form a shrewd guess of the probable amount of correction. (The effect of a possible error will be exhibited hereafter.) We have now all the six elements required for the investigation.

Let moon's corrected altitude + moon's app. alt. $= 2A$;
moon's corrected altitude − moon's app. alt. $= 2a$;
sun's apparent altitude + sun's corr. alt. $= 2B$;
sun's apparent altitude − sun's corr. alt. $= 2b$;
corrected distance + apparent distance $= 2C$;
corrected distance − apparent distance $= 2c$.

* Eigentlich Methode von Legendre; cfr. Weyer, Ann. d. Hydrographie 1882, pag 344 W.

Then, moon's apparent altitude $= A - a$; corr. alt. $= A + a$;
sun's apparent altitude $= B + b$; corr. alt. $= B - b$;
apparent distance $= C - c$; corr. distance $= C + c$.

The essential circumstance which directs the further investigations is the equality of the zenithal angles, and consequently of the cosines of the zenithal angles. The corresponding equation is—

$$\frac{\cos \overline{C-c} - \sin.\,\overline{A-a}\ \sin.\,\overline{B+b}}{\cos \overline{A-a}\,.\,\cos \overline{B+b}} = \frac{\cos.\,\overline{C+c} - \sin.\,\overline{A+a}\,.\,\sin.\,\overline{B-b}}{\cos \overline{A+a}\,.\,\cos \overline{B-b}};$$

or, multiplying out the denominators—

$$\text{(First side) } \cos.\,\overline{C-c}\,.\,\cos.\,\overline{A+a}\,.\,\cos.\,\overline{B-b}$$
$$-\sin.\,\overline{A-a}\ \sin.\,\overline{B+b}\,.\,\cos.\,\overline{A+a}\,.\,\cos.\,\overline{B-b}$$
$$\text{(Second side} = \cos.\,\overline{C+c}\,.\,\cos.\,\overline{A-a}\,.\,\cos.\,\overline{B+b}$$
$$-\sin.\,\overline{A+a}\ \sin.\,\overline{B-b}\,.\,\cos.\,\overline{A-a}\,.\,\cos.\,\overline{B+b}\,.$$

For development of these terms, it must be remembered that—

$$\sin.\,\overline{A+a} = \sin.\,A\,.\,\cos.\,a + \cos.\,A\,.\,\sin.\,a,$$
$$\sin.\,\overline{A-a} = \sin.\,A\,.\,\cos.\,a - \cos.\,A\,.\,\sin.\,a;$$
$$\cos.\,\overline{A+a} = \cos.\,A\,.\,\cos.\,a - \sin.\,A\,.\,\sin.\,a,$$
$$\cos.\,\overline{A-a} = \cos.\,A\,.\,\cos.\,a + \sin.\,A\,.\,\sin.\,a;$$

and similarly for B and C.

We now proceed to develop the first side.

Making the substitutions just stated, the first large product gives—

Line 1, not containing sin. a, or sin. b, or sin. c—

$$+\cos.\,A\,.\cos.\,B\,.\cos.\,C\,.\cos.\,a\,.\cos.\,b\,.\cos.\,c.$$

Line 2, containing simply sin. a, or sin. b, or sin. c—

$$+\cos.\,A\,.\cos.\,B\,.\sin.\,C\,.\cos.\,a\,.\cos.\,b\,.\sin.\,c.$$
$$-\sin.\,A\,.\cos.\,B\,.\cos.\,C\,.\sin.\,a\,.\cos.\,b\,.\cos.\,c.$$
$$+\cos.\,A\,.\sin.\,B\,.\cos.\,C\,.\cos.\,a\,.\sin.\,b\,.\cos.\,c.$$

Line 3, containing sin. a . sin. b, or sin. b . sin. c, or sin. a . sin. c—

$$-\sin.\,A\,.\sin.\,B\,.\cos.\,C\,.\sin.\,a\,.\sin.\,b\,.\cos.\,c.$$
$$+\cos.\,A\,.\sin.\,B\,.\sin.\,C\,.\cos.\,a\,.\sin.\,b\,.\sin.\,c.$$
$$-\sin.\,A\,.\cos.\,B\,.\sin.\,C\,.\sin.\,a\,.\cos.\,b\,.\sin.\,c.$$

Line 4, containing sin. a . sin. b . sin. c—

$$-\sin.\,A\,.\sin.\,B\,.\sin.\,C\,.\sin.\,a\,.\sin.\,b\,.\sin.\,c.$$

And the second large product gives—

Line 1, $-\sin.\,A\,.\cos.\,A\,.\sin.\,B\,.\cos.\,B.$

Line 2, $-\sin. A . \cos. A . \sin. b . \cos. b + \sin. B . \cos. B . \sin a . \cos. a$.

Line 3, $+\sin. a . \cos. a . \sin. b . \cos. b$.

There is no line 4.

We now examine the second side of the equation.

The difference between the first side and the second side is this: that in every place where a occurs in the equation, or sin. a in the development, of the first side, $-a$ or $-\sin. a$ occurs on the second side; and similarly for b, sin. b, c, sin. c. And, moreover, these changes occur simultaneously; so that wherever $\sin. a \times \sin. b$ occurs on the first side there will be $-\sin. a \times -\sin. b$ on the second side; and where $\sin. a \times \sin. b \times \sin. c$ occurs on the first side there will be $-\sin. a \times -b \times -\sin. c$ on the second side. And thus we see that—

For line 1 the first side and the second side are the same.

For line 2 the first side and the second side are equal, but have opposite signs.

For line 3 the first side and the second side are the same.

For line 4 the first side and the second side are equal, but have opposite signs.

Therefore, transferring the second side with sign changed to the first side, the equation becomes the following:—

$$\left.\begin{cases} +2 . \cos. A . \cos. B . \sin. C . \cos. a . \cos. b . \sin. c \\ -2 \sin. A . \cos. B . \cos. C . \sin. a . \cos. b . \cos. c \\ +2 . \cos. A . \sin. B . \cos. C . \cos. a . \sin. b . \cos. c \\ -2 . \sin. A . \sin. B . \sin. C . \sin. a . \sin. b . \sin. c \\ -2 \sin. A . \cos. A . \sin. b . \cos. b \\ +2 . \sin. B . \cos. B . \sin. a . \cos. a \end{cases}\right\} = 0.$$

This equation is rigorously accurate.

We will now consider what simplification it will admit, preserving the character of practical accuracy of the highest order.

a, which is half the correction of the moon's altitude, can never exceed $30'$. Cosine a can never differ from 1 by $\frac{1}{20000}$ part, and $\frac{\sin. a}{a}$ can never differ from 1 by $\frac{1}{60000}$ part; and therefore for cos. a and sin. a we may put 1 and a. The same applies to b and c. In the product $\sin. a . \sin. b . \sin. c$, the factor of sin. c can rarely or never amount to $\frac{1}{50000}$, and that term may be neglected. The equation now becomes—

$$\left\{\begin{array}{l} +\cos. A\ .\ \cos. B\ .\ \sin. C \times 2c \\ -\sin. A . \cos. B . \cos. C \times 2a + \cos. A . \sin. B . \cos. C \times 2b \\ -\sin. A\ .\ \cos. A \times 2b + \sin. B\ .\ \cos. B \times 2a \end{array}\right\} = 0$$

Remarking that $2a$, $2b$, $2c$ are the corrections of moon's altitude, sun's altitude, and distance, the result of this equation is—

$$\begin{array}{c}\text{Corr. of} \\ \text{dist.}\end{array}\left\{\begin{array}{c} (+\tan. A\ .\ \text{cotan. } C - \sec. A\ .\ \sin. B\ .\ \text{cosec. } C) \\ \times \text{ correction of moon's altitude} \\ (-\tan. B\ .\ \text{cotan. } C + \sin. A\ .\ \sec. B\ .\ \text{cosec. } C) \\ \times \text{ correction of sun's altitude} \end{array}\right\}.$$

The only opening to error in this formula is in the estimated value of C, as depending on error in the estimated "Nautical Almanac" distance, or in the estimated correction to the observed distance. Suppose that the time by account was 4m. in error (implying error of 1° in longitude). The approximate correction of distance would be taken out about $2'$ in error, and C would be about $1'$ in error. If the value of the distance was about 60°, an error of $1'$ would produce in cotan. C an error of about $\frac{1}{1400}$ of that term of the computed correction, and in cosec. C the error would be $\frac{1}{6000}$. These would be hardly sensible. But if, with C corrected by this approximation, the calculation be repeated (requiring only a few minutes), the error of result will be totally insensible.

The following is offered as a form proper to be used with this method:—

Prepare this table, inserting numbers instead of the printed words.

Correct. to moon's apparent alt. (additive).	Correct. to sun's apparent alt. (subtractive).	First Approximation.	Second approximation (if necessary).
		Assumed cor. to app. distance (additive) (sub.).	Assumed cor. to app. distance (additive) (sub.).
Moon's app. alt. Moon's cor. alt.	Sun's app. alt. Sun's cor. alt.	App. distance. Cor. distance.	App. distance. Cor. distance.
Sum A = Half sum.	Sum B = Half sum.	Sum C = Half sum.	Sum C = Half sum.

Then proceed with the following calculations, using 5-figure-logarithms—

First Approximation.	Second Approximation (if necessary).
Additive Terms.	Additive Terms.
Log. tan A	(Repeat) log. tan. A ....
Log. cotan. C	Log. cotan. C ....
Log.cor. to moon's alt.	(Rep.) log. cor.to moon's alt.
Sum and number.......	Sum and number...............
Log. sine A	(Repeat) log. sine A ....
Log. secant B	(Repeat) log. secant B ......
Log. cosecant C	Log. cosecant C ....
Log. cor. to sun's alt.	(Rep.) log. cor. to sun's alt.
Sum and number.......	Sum and number..............
Sum of additive terms	Sum of additive terms ......
Subtractive Terms.	Subtractive Terms.
Log. secant A	(Repeat) log. secant A ......
Log. sine B	(Repeat) log. sine B ....
Log. cosecant C	Log. cosecant C ....
Log. cor.to moon's alt.	(Rep.) log.cor. to moon's alt.
Sum and number	Sum and number................
Log. tan. B	(Repeat) log. tan. B ....
Log. cotan. C	Log. cotan. C ....
Log. cor. to sun's alt.	(Rep.) log. cor. to sun's alt.
Sum and number	Sum and number...............
Sum of subtract. terms	Sum of subtractive terms...
Combination of additive and subtractive = correction to apparent distance	Combination of additive and subtractive = correction to apparent distance ....

This form supposes that C is less than 90°. When C exceeds 90°, the supplement to 180° is to be taken, the cosecant and cotangent of that supplement are to be used, and the signs of the first and fourth numbers, which are produced by cotan. C, are to be changed; the first number will become subtractive, and the fourth number additive.

The second approximation will very rarely be required. If, however, the final "correction to apparent distance" differ from that assumed at the beginning by 2′ or 3′, it may be satisfactory to use the second approximation; it is very easy.

The true distance found by either method is for a sphere, and, therefore, to adapt it to the spheroidal form of the earth, the altitudes should be corrected for the angle of the vertical before using them in clearing the distance. Rules for that purpose are found on p. 65.

RULE (a) Compute the azimuth, and if no tables of the angle of the vertical are at hand, compute also its value.

(b) Then :—Cor. for alt. = angle of vert. × cos. azimuth.

(c) In calculating the apparent altitude from the true, subtract the correction in (b) from the true altitude before correcting it for parallax if the azimuth is greater than 90°; but add the correction in (b) to the true altitude when the azimuth is less than 90°. The result is the true altitude reduced to the terrestrial spheroid.

But as the exact altitudes are not required in clearing the distance, and as errors in longitude arising from instrumental errors, or errors in observation, are probably greater than those arising from altitudes, the correction for the spheroidal form of the earth is seldom used at sea.

Ex. 522. With the following data compute the true distance between the moon and a star.

Apparent altitudes.	*True altitudes.*	*Apparent distance.*
m' 21° 24′ 46″	m 22° 18′ 6″	d' 57° 30′ 22·6″
s' 67 13 8	s 67 12 44	

In order to avoid the necessity for taking out the logarithms to seconds, we shall add 14″ to each of the moon's altitudes, subtract 8″ from each of the star's altitudes, and reject 22·6″ from the apparent distance in solving this question by the first and third methods. The data will then stand thus :—

Apparent altitudes.	*True altitudes.*	*Apparent distance.*
m' 21° 25′ 0″	m 22° 18′ 20″	d' 57° 30′ 0″
s' 67 13 0	s 67 12 36	

BORDA'S METHOD :—

$$\left.\begin{aligned} \cos.\,\theta &= \sqrt{\sec.\,m'\,.\,\sec.\,s'\,.\,\cos.\,S\,.\,\cos.\,(S-d')\,.\,\cos.\,m\,.\,\cos.\,s} \\ \phi &= \frac{m+s}{2} \\ \sin.\,\frac{d}{2} &= \sqrt{\sin.\,(\theta+\phi)\,.\,\sin.\,(\theta-\phi)} \end{aligned}\right\}$$

d'	57°	30′	0″		
m'	21	25	0	sec.	·031074
s'	67	13	0	sec.	·412011
	2)146	8	0		
S	73	4	0	cos.	9·464279
$S - d'$	15	34	0	cos.	9·983770
m	22	18	20	cos.	9·966223
s	67	12	36	cos.	9·588109
ϕ	44	45	28		2)19·445466
θ	58	7	17	cos.	9·722733
Sum	102	52	45	sin.	9·988934
Diff.	13	21	49	sin.	9·363857
					2)19·352791
$\frac{d}{2}$	28°	20′	17·3″	sin.	9·676395
			2		
d	56	40	34·6		
Add		+	22·6	rejected before	
True distance	56	40	57·2		

KRAFFT'S METHOD :—

$$\cos. A = \tfrac{1}{2} \cos. m \,.\, \sec. m' \,.\, \cos. s \,.\, \sec. s'.$$

m	22°	18′	20″	cos.	1·966223
m'	21	25	0	sec.	·031074
s	67	12	36	cos.	9·588109
s'	67	13	0	sec.	·412011
					9·997417
				2 log.	·301030
A	60°	11′	45·5″	cos.	9·696387
d'	57	30	22·6		
$m' + s'$	88	37	54		
$Z + z$	90	29	10		

$$Vers.\ d = vers.\ (sum\ true\ z.\ d.)\ .\ vers.\ (d' + A)\ .\ vers.\ (d' \backsim A)$$
$$vers.\ \{(m' + s') + A\}\ .\ vers.\ \{(m' + s') \backsim A\}.$$

$Z + z$	90° 29′ 10″	vers. 1·008436	+	48
$d' + A$	117 42 8	vers. 1·464842	+	34
$d' \backsim A$	2 41 23	vers. ·001096	+	5
$(m' + s') + A$	148 49 40	vers. 1·855515	+	101
$(m' + s') \backsim A$	28 26 8	vers. ·120628	+	18
	Correction for seconds	206		206
True distance	56 40 57·5	vers. 4·450723		

MERRIFIELD'S METHOD:—

C = correction for moon's altitude $= m - m'$
$= 53'\ 20'' = 3200$;
c = correction for star's altitude $= s' - s$
$= 24''$.

True distance $= d' - (C - c) + 2$ cosec. d' . cos. S . $(M - N)$.

m'	21° 25′ 0″	sec. ·031074		
s'	67 13 0		sec. ·412011	
d'	57 30 0			cosec. ·073971
	2)146 8 0			
S	73 4 0			cos. 2·464279
$S - s'$	5 51 0	sin. 9·008278		
$S - m'$	51 39 0		sin. 9·894446	
C	53′20″ = 3200	log. 3·505150		
c	24		log. 1·380211	
M	350·35	log. 2·544502		
N	48·6		log. 1·686668	
$M - N$	301·75			log. 2·479647

$$C\ .\ \sin.^2 \frac{\theta}{2} + c\ .\ \sin.^2 \frac{\phi}{2} = 104{\cdot}2'' \qquad \log.\ 2{\cdot}017897$$

	= + 1′ 44·2″
	2
	+ 3 28 4
$C - c$	− 52 56
Correction for dist.	− 49 27·6
App. cent. dist.	57 30 22·6
True distance	57 40 55

Precautions to be taken.—(1) Three persons should be engaged in taking a lunar; the best observer always measuring the distance, which must be done with the utmost accuracy, because an error in distance will, on the average, produce thirty-three times its amount in the deduced longitude: thus 10″ error in distance will produce 330″ or $5\frac{1}{2}'$ error in longitude. The other two persons should observe the altitudes.

(2) Sets of four or five simultaneous observations should be made in quick succession, and the means used instead of a single one, because the average of several is probably more correct than any one observation.

(3) The altitudes need not be taken so exact as the distance. The reason for this is, that refraction and parallax, which are the only causes of a difference between the apparent and true distances, vary very slowly for medium altitudes: hence a small error in altitude will not affect the amount of these corrections to any appreciable extent, and therefore will not affect the true distance. This being so, although the Greenwich time may be considerably in error, still the true distance may be obtained with great accuracy when the altitudes are calculated. For star lunars the practical navigator prefers to calculate the altitudes.

(4) The moon's bright limb is always turned towards the sun; hence if a sun lunar be taken the nearer limb is always observed, but with a star lunar it must be carefully noted which limb is used, so that in reducing the observed to the apparent distance it may be known whether the moon's semidiameter is to be added or subtracted to find the apparent central distance of the star.

(5) If neither of the objects observed for a lunar distance be in such a position as to justify its use in obtaining local time, this can be calculated either before or after from some other object suitably situated for that purpose; and if the course and distance in the interval elapsed between the observations be noted and allowed for, the result will be as exact as that obtained by altitudes taken at the instant of observing the distance.

(6) Greenwich mean time from lunars should be deduced from distances of objects as nearly equal as possible on different sides of the moon, and should be taken by the same person with the same instrument, shades, and telescope, because the errors of observation will then most nearly neutralize each other; but the mean of Greenwich times thus calculated will not be the correct one unless the moon changes her distance from both bodies at the same rate, otherwise the correct time will be the instant intermediate proportional to the rates.

(7) Captain Toynber recommends that errors in longitude for sun and star lunars, both east and west of the moon, should in each case be tabulated, as the personal errors of the observer, and applied to all observations made.

Checks to Work.—(*a*) The student should bear in mind that the true can never differ from the apparent distance by more than the sum of the corrections for the altitudes of the objects, which rarely exceeds a degree.

(*b*) When the moon's altitude is equal to or less than that of the other body, the true distance is less than the apparent distance, and *vice versâ.*

(*c*) When the apparent distance is greater than 90°, it is almost always greater than the true distance.

Objections to Lunars.—(1) The great care required in measuring the distance, because of the slow motion of the moon among the heavenly bodies, and the large errors which result from an inaccurate distance.

(2) The great length of the computation.

But when we consider the check which lunars properly selected are on the performance of the chronometer, neither of these objections should deter the careful mariner from practising them as often as practicable at sea.

The following questions are worked in full to show the student how he should arrange his work.

Ex. 523. 1887. Nov. 20th, at about 3h. 13m. 40s. mean time at ship, in latitude 41° 56′ S., longitude 38° 49′ E., the observed altitude of the sun's L. L. was 40° 50′ 10″, index error + 2′ 14″; the observed altitude of the moon's L. L. was 59° 54′ 40″, index error + 5′ 15″, and the observed distance between the enlightened limbs of the sun and moon was 67° 39′ 50″, index error − 1′ 17″, height of the eye 22 feet. Find the longitude.

For Greenwich mean time.

Mean time at ship, Nov.	20d.	3h.	13m.	40s.
Long., E.	−	2	35	16
G. M. time,	Nov. 20	0	38	24

Long. in time.

Long. 38° 49′ E.
In time 2h. 35m. 16s.

Sun's declination.

November 20	19° 41′ 57·0″ S.
Correction	+ 21·8
True declination	19 42 18·8 S.
S. P. D.	70 17 41·2

Variation of declination.

In 1 hour	+ 34·08″
No. hours	·64
Correction	21·8112

Equation of time.		*Variation of eq. time.*	
November 20	14m. 15·04s.	In 1 hour	− ·603s.
Correction	− 0·39	No. hours	·64
True eq. time	+ 14 14·65 to M. T.	Correction	− ·38592

Moon's semidiameter.		*Variation for semidiameter.*	
Nov. 20, at noon	15′ 29·2″	In 12 hours	− 7·0″
Correction	− ·4	No. hours	·64
	15 28·8		12)4·48
Augmentation	+ 13·4	Correction	− ·37
True semidiameter	15 42·2		

Moon's horizontal parallax.		*Variation for hor. parallax.*	
Nov. 20, at noon	56′ 44·4″	In 12 hours	− 25·7″
Correction	− 1·4	No. hours	·64
	56 43		12)16·448
Reduction	− 5	Correction	− 1·37
True hor. parallax	56 38 = 3398″		

For moon's R. A.		*Variation of R. A.*	
November 20, 0h.	20h. 30m. 28·57s.	In 10m.	+ 22·475
Correction	+ 1 26·30	No. of 10m	3·84
True R. A. moon	20 31 54·87	Correction	+ 86·30400

For moon's declination.		*Variation of declination.*	
Nov. 20, 0h.	18° 17′ 32·1″ S.	In 10m.	− 50·97
Correction	− 3 15·7	No. of 10m.	3·84
True dec. moon	18 14 16·4 S.	Correction	− 195·7248
S. P. D.	71 45 43·6		

To correct sun's alt.		*To correct moon's alt.*		
Obs. alt. L.L.	40° 50′ 10″	Obs. alt. L.L.	59° 54′ 40″	
Index error	+ 2 14	Index error	+ 5 15	
	40 52 24		59 59 55	
Dip	− 4 37	Dip	− 4 37	
	40 47 47		59 55 18	
Semidiameter	+ 16 13·9	Semidiam.	+ 15 42·2	
App. alt. s'	41 4 0·9	App. alt. m'	60 11 0·2	. . cos. 9·696553
Refraction	− 1 5	Refraction	− 33	H.P. 3398 log. 3·531223
	41 2 55·9		60 10 27·2	
Par. in alt.	+ 6	Par. in alt.	+ 28 10	= 1690 log. 3·227776
True alt. s	41 3 2	True alt. m	60 38 37	

For apparent central distance.

Observed dist. nearest limbs	67° 39′ 50″
Ind. error	− 1 17
	67 38 33
Sun's semidiameter	16 13·9
Moon's semidiameter	15 42·2
Apparent central distance	68 10 29

To clear the distance.

I. BORDA'S METHOD:—

$$\left.\begin{aligned} \cos.\theta &= \sqrt{\sec. m'.\sec. s'.\cos. S.\cos.(S-d')\cos. m.\cos. s} \\ \phi &= \frac{m+s}{2} \\ \sin.\frac{d}{2} &= \sqrt{\sin.(\theta+\phi).\sin.(\theta-\phi)} \end{aligned}\right\}$$

d	68°	11′	0″		
m'	60	11	0	sec.	·303446
s'	41	4	0	sec.	·122660
	2)169	26	0		
S	84	43	0	cos.	8·964170
$S-d'$	16	32	0	cos.	9·981662
m	60	38	37	cos.	9·690410
s	41	3	1	cos.	9·877448
ϕ	50	50	49		2)18·939796
θ	72	50	22	cos.	9·469898
$\theta+\phi$	123	41	11	sin.	9·920169
$\theta-\phi$	21	59	33	sin.	9·573435
					2)19·493604
$\frac{d}{2}$	33	55	56·9	sin.	9·746802
d	67	51	53·8		
Subtract			31		
True distance	67	51	22·8		

II. KRAFFT'S METHOD :—

$$\cos. A = \tfrac{1}{2} \cos. m \, . \sec. m' \, . \cos. s \, . \sec. s'.$$

m	60°	38′	37″	cos.	9·690410
m'	60	11	0	sec.	·303446
s	41	3	2	cos.	9·877446
s	41	4	1	sec.	·122662
					9·993964
				2 log.	·301030
A	60°	27′	20″	cos.	9·692934
d'	68	10	29		
$m'+s'$	101	15	1		
$Z+z$	78	18	21		

$$Vers. d = vers. (sum\ true\ z.\ d.) \, . \, vers. (d' + A) \, . \, \{vers. d' \backsim A\} \, . \, vers. \{(m' + s') + A\} \, . \, vers. \{(m' + s') \backsim A\}.$$

$Z + z$	78°	18′	21″	vers.	·797213	+ 100
$d' + A$	128	37	49	vers.	1·624107	+ 186
$d' \backsim A$	7	43	9	vers.	·009056	+ 6
$(m' + s') + A$	161	42	21	vers.	1·949426	+ 32
$(m' + s') \backsim A$	40	47	41	vers.	·242815	+ 129
	Parts for seconds				453	453
True distance	67	51	23	vers.	4·623070	

III. MERRIFIELD'S METHOD :—

C = correction for moon's alt. = 27′ − 37″
= 1657″;

c = correction for sun's alt.
= 59″.

True distance $= d' - (C - c) + 2\,\text{cosec.}\,d'.\,\cos.\,S\,(M - N)$.

m'	60° 11′ 0″	sec. ·303446		
s'	41 4 0		sec. ·122660	
d'	68 11 0			cosec. ·032275
	2)169 26 0			
S	84 43 0			cos. 8·964170
$S - s'$	43 39 0	sin. 9·839007		
$S - m'$	24 32 0		sin. 9·618281	
C	27′37″ = 1657	log. 3·219323		
c	59 = 59		log. 1·770852	
M	2300·3	log. 3·361776		
N	32·5	. . . . . .	log. 1·511793	
$M - N$	2267·8	. . . . . .	. . . .	log. 3·355467

$C\,.\,\sin.^2\frac{\theta}{2} - c\,.\,\sin.^2\frac{\phi}{2} = 224·9$ log. 2·351912

	=	3′	44·9″
			2
	+	7	29·8
$C - c$	−	26	38
	−	19	8·2
d'	68	10	29
True distance	67	51	20·8

For Greenwich mean time.

True distance	67°	51′	21″	
Dist. at noon	67	32	19	P. L. 3056
		19	2	P.L. 9758
Approximate interval	0h.	38m.	28s.	P. L. 6702
Time from N. A.	0	0	0	
Approximate G. M. T.	0	38	28	
Cor. for 2nd diff.		−	4	
G. M. T. Nov. 20	0	38	24	

For time at ship and longitude.

	⊙ alt	41°	3′	2″		
φ	lat	41	56	0	sec.	·128472
$(90-\delta)=$ ⊙	pol.dist	70	17	41	cosec.	·026208
	2)153		16	43		
	S	76	38	21·5	cos.	9·363763
	$S-a$	35	35	19·5	sin.	9·764895
						2)19·283338
	$\frac{h}{2}$	25	59	19·7	sin.	9·641669
				2		
Westerly hr. angle ⊙		51	58	39·4		
				4		
	60)207		54	37·6		

App. time ship, Nov. 20	3h.	27m.	54·63s.
Eq. time	−	14	14·65
Mean time ship, Nov. 20	3	13	40
Mean time Gr., Nov. 20	0	38	24
Longitude	2	35	16 = 38° 49′ 0″ E.

Ex. 524. 1887, July 7th, p.m. at ship in latitude 48° 23′ S., with the following observations, find the true and apparent altitudes of Jupiter, the error of the chronometer for Greenwich mean time, and the longitude of the place :—

Approx. M. T. G. by chro.	*Alt. moon's U.L.*	*Obs. dist. furthest limbs.*
7d. 20h. 19m. 30s.	18° 46′ 40″	115° 50′ 0″
	I. E. + 1 35	I. E. + 3 8

The height of the eye was 18 feet, moon east of meridian.

For moon's semidiameter.		*Variation of semidiameter.*	
July 7 at 12h.	15′ 10·2″	In 12 hours	− 4.4″
Correction	− 3·1	No. hours	8·325
	15 7·1		12)36·6300
Augmentation	+ 4·8	Correction	− 3·1
True semidiameter	15 11·9		

For moon's horizontal parallax.

July 7 at 12h.	55′ 34·6″
Correction	− 11·1
	55 23·5
Reduction	− 6·2
True hor. parallax	55 17·3 = 3317·3″

Variation of hor. parallax.

In 12 hours	− 16·0″
	8·325
	12)133·200
Correction	− 11·1

For moon's R. A.

July 7, 20h.	21h. 34m.		4·57s.
Correction		+	40·62
True R. A. moon	21	34	45·19

Variation of R. A.

In 10m.	+ 20·830s.
No. of 10m.	1·95
Correction	40·61850

For moon's declination.

July 7, 20h.	14° 38′ 23·1″ S.
Correction	− 2 21·0
True declination	14 36 2·1 S.
S. P. D.	75 23 58

Variation of declination.

In 10m.	− 72·33
No. of 10m.	1·95
Correction	141·0435

For R. A. Jupiter.

7 July	13h.	39m.	17·29s.
8 July	13	39	26·74
Variation in 24h.		+	9·45
„ 20·325		+	8·00
R. A. on 7th	13	39	17·29
True R. A.	13	39	25·29

For declination of Jupiter.

7 July	9°	0′	17·7″	S.
8 July	9	1	29·3	S.
Variation in 24h.	+	1	11·6	
„ 20·325	+	1	0·6	
Declination on 7th	9	0	17·7	S.
True declination	9	1	18·3	S.
S. P. D.	80	58	41·7	

For R. A. of mean sun.

Sidereal time, July 7	7h.	0m.	32·79s.
Acceleration for { 20h.		3	17·13
19m.			3·12
30s.			·08
R. A. mean sun	7	3	53·12

For mean time at place.

Obs. alt. moon's U. L.	18°	46′	40″	
Index error	+	1	33	
	18	48	13	
Dip	−	4	11	
	18	44	2	
Semidiameter	−	15	12	
App. alt. moon m'	18	28	50	. . . . cos. 9·977006
Refraction	−	2	47	
	18	26	3	Hor. par. 3317·3″ log. 3·520784
Par. in alt.	+	52	26	= 3146″ log. 3·497790
True alt. moon m	19	18	29	
Latitude	48	23	0	sec. ·177738
Polar distance	75	23	58	cosec. ·014256
	2)143	5	27	
S	71	32	43·5	cos. 9·500447
$S - a$	52	14	14·5	sin. 9·897932
				2)19·590373
$\frac{h}{2}$	38°	36′	32·4″	sin. 9·795186
			2	
Eastern hour angle	77	13	4·8	
			4	
	60)308	52	19·2	
Eastern hour angle	5h.	8m.	52·32s.	
	24			
Moon's western hr. angle	18	51	7·68	
Moon's right ascension	21	34	45·19	
R. A. meridian	16	25	52·87	
R. A. mean sun	7	3	53·12	
Mean time at place	9	21	59·75	

To compute Jupiter's altitude.

R. A. meridian	16h.	25m.	52·87s.
R. A. Jupiter	13	39	25·29
Western hour angle	2	46	27·58
h In arc	41°	36′	54″

$\therefore \frac{h}{2}$	20°	48′	27″	cos.	9·970709
					2
				cos.²	19·941418
l'	41	37	0	sin.	9·822262
p	80	58	42	sin.	9·994594
$\frac{l'+p}{2}$	61	17	51		2)19·758274
θ	49	12	24	sin.	9·879137
Sum	110	30	15	sin.	9·971576
Diff.	12	5	27	sin.	9·321105
					2)19·292681
Half z. d.	26°	17′	29″	sin.	9·646340
			2		
Zen. dist.	52	34	58		
	90				
Jupiter's true alt.	37	25	2 (s)		
Par. in alt.		−	1		
	37	25	1		
Refraction	+	1	14		
Jupiter's app. alt.	37	26	15 (s')		

For apparent central dist.

Observed dist.	115°	50′	0″
Index error	+	3	8
	115	53	8
Moon's S.D. 15′ 12″ } Jupiter's S.D. 18 }	−	15	30
App. cent. dist.	115	37	38

To clear the distance.

I. BORDA'S METHOD :—

$$\left.\begin{aligned}\cos.\theta &= \sqrt{\sec.m'.\sec.s'.\cos.S.\cos.(S-d')\cos.m.\cos.s}\\ \phi &= \frac{m+s}{2}\\ \sin.\frac{d}{2} &= \sqrt{\sin.(\theta+\phi).\sin.(\theta-\phi)}\end{aligned}\right\}$$

d'	115°	37′	0″		
m'	18	29	0	sec.	·023001
s'	37	26	0	sec.	·100146
	2)171	32	0		
S	85	46	0	cos.	8·868165
$S-d'$	29	51	0	cos.	9·938185
m	19	18	37	cos.	9·974851
s	37	24	47	cos.	9·899971
ϕ	28	21	42		2)18·804319
θ	75	22	41	cos.	9·402160
Sum	103	44	23	sin.	9·987391
Diff.	47	0	59	sin.	9·864243
					2)19·851634
$\frac{d}{2}$	57°	27′	22″	sin.	9·925817
			2		
d	114	54	44		
Add			38	rejected before.	
True distance	114	55	22		

II. KRAFFT'S METHOD :—

$$\cos.A = \tfrac{1}{2}\cos.m.\sec.m'.\cos.s.\sec.s'.$$

m	19°	18′	29″	cos.	9·974859
m'	18	28	50	sec.	·022994
s	37	25	2	cos.	9·899948
s'	37	26	15	sec.	·100170
					9·997971
				2 log.	·300 30
A	60	9	14	cos.	9·696941
d'	115	37	38		
$m'+s'$	55	55	7		
$Z+z$	123	16	29		

Vers. d = *vers.* (*sum true z. d.*) . *vers.* $(d' + A)$. vers. $(d' \backsim A)$
vers. $\{(m' + s') + A\}$. *vers.* $\{(m' + s') \backsim A\}$.

$Z + z$	123°	16′	29″	vers.	1·548537	+	118
$d' + A$	175	46	52	vers.	1·997272	+	20
$d' \backsim A$	55	28	24	vers.	·433114	+	95
$(m' + s') + A$	116	4	21	vers.	1·439417	+	92
$(m' + s) \backsim A$	4	14	7	vers.	·002728	+	2
	Correction for seconds				327		327
True distance	114	55	22	vers.	1·421395		

III. Merrifield's Method :—

C = correction for moon's altitude = 49′ 37″
= 2977″;
c = correction for planet's altitude = 1′ 13″
= 73″.

True distance = $d' - (C - c) + 2$ *cosec.* d'. *cos.* $S\,(M - N)$.

m'	18°	29′	0″	sec. ·023001					
s'	37	26	0			sec. ·100146			
d'	115	37	0					cosec.	·044935
	2)171	32	0						
S	85	46	0					cos.	8·868165
$S - s'$	48	20	0	sin. 9·873335					
$S - m'$	67	17	0			sin. 9·964931			
C	49′37″ = 2977			log. 3·473779					
c	1·13 = 73					log. 1·863323			
M	2344·8			log. 3·370115					
N	84·8			. . . . . .		log. 1·928400			
$M - N$	2260			. . . .				log.	3·354108

$C \cdot \sin.^2 \frac{\theta}{2} + c \cdot \sin.^2 \frac{\phi}{2} = 185{\cdot}0''$ log. 2·267208
= + 3′ 5″
2

	+	6	10
$C - c$	−	48	24
Correction for distance	−	42	14
Apparent cent. distance	115	37	38
True distance	114	55	24

For Greenwich mean time.

True distance	114°	55′	24″			
Dist. at XVIII.	113	43	26	P. L. 2880		
	1	11	58	P. L. 3981		
Approximate interval	2h.	19m.	41s.	P. L. 1101		
Time from N. A.	18	0	0			
Approximate G.M.T.	20	19	41			
Cor. for 2nd diff.		—	3			
G. M. T. July 7	20	19	38			

For longitude.

Greenwich mean time, July	7d.	20h.	19m.	38s.
Ship „ „ July	7	9	21	59·75
Longitude in time		10	57	38·25
	=	164°	24′	34″ W.

For error of chronometer.

Greenwich mean time, July	7d.	20h.	19m.	38s.
Chronometer showed, July	7	20	19	30
Chronometer slow				8

True altitude Jupiter 37° 25′ 2″
App. „ „ 37 26 15
Error of chro. for G. M. T. 8 secs. slow
Longitude of place 164° 24′ 34″ W.
} *Answer.*

EXERCISE XXIV.

Ex. 525. 1887, February 13th, at about 8h. 3m. a.m. apparent time at ship, in latitude by account 12° 18′ N., longitude 126° 47′ W., the observed altitude of the sun's L. L. was 25° 45′ 30″; index error − 2′ 11″; the observed altitude of the moon's L. L. was 33° 43′ 0″; index error + 2′ 57″; and the observed distance between the sun and moon's enlightened limbs was 107° 30′ 30″; index error − 3′ 14″; height of the eye 19 feet. Required the longitude.

Ex. 526. 1887, October 3rd, at about 4h. 37m. a.m. mean time at ship, in latitude by account 14° 39′ S., longitude

42° 51′ E., the observed altitude of the star Aldebaran was 56° 30′ 20″; index error − 2′ 5″; the observed altitude of the moon's U. L. was 26° 30′ 30″; index error − 3′ 0″; the observed distance between the moon's nearest limb and the star was 48° 55′; index error + 38″; height of the eye 23 feet. Find the longitude.

Ex. 527. 1887, August 14th, mean time at ship about 10h. 40m. a.m., in latitude by account 27° 20′ N., longitude 93° 20′ 45″ W., the observed altitude of the moon's L. L. was 51° 30′ 20″; index error + 13″; the observed altitude of the sun's U. L. was 66° 47′ 0″; index error − 26″; height of the eye 21 feet. The observed distance between the sun and moon was 58° 40′ 0″; index error − 39″. Required the longitude.

Ex. 528. 1887, February 8th, a.m. at ship in latitude 19° 52′ S., longitude about 141° 15′ E., with the following observations find the true and apparent altitudes of the moon, the error of the chronometer for Greenwich mean time, and the longitude of the place:—

Approx. M. T. G. by chro.	*Alt. Pollux.*	*Obs. dist. nearest limbs.*
5h. 23m. 15s.	10° 59′ 30″	17° 58′ 40″
	I. E. + 3 12	− 1 28

The height of the eye was 18 feet.

Ex. 529. 1887, April 5th, p.m. at ship in latitude 18° 42′ N., longitude about 136° 20′ E., with the following observations find the true and apparent altitude of Venus, the error of the chronometer for Greenwich mean time, and the longitude of the place:—

Approx. M. T. G. by chro.	*Alt. moon's L. L.*	*Obs. dist. nearest limbs.*
10h. 27m. 40s. a.m.	57° 51′ 15″	111° 29′ 40″
	I. E. + 2 25	I. E. − 2 4

The height of the eye 27 feet.

Ex. 530. 1887, May 24th, p.m. at ship in latitude 33° 36′ N., longitude about 30° 50′ E., with the following observations find the true and apparent altitudes of the moon, the error of the chronometer for Greenwich mean time, and the longitude of the place:—

Approx. M. T. G. by chro.	*Alt. sun's L. L.*	*Obs. dist. nearest limbs.*
1h. 17m. 24s. p.m.	43° 9′ 0″	18° 28′ 20″
	I. E. − 1 2	I. E. − 5

Height of the eye 19 feet.

Ex. 531. 1887, September 4th, at about 3h. 58m. 43s. a.m. mean time at ship, in latitude 27° 45′ S., longitude 166° 20′ E., the observed altitude of the moon's L. L. was 39° 57′ 40″; index error − 5′ 11″; the observed altitude of Fomalhaut 38° 41′ 15′; index error + 3′ 26″; height of the eye 23 feet. The observed distance between the star and the moon's farthest limb was 28° 25′ 50″; index error + 3′ 3″. Find the longitude.

Ex. 532. 1887, December 30th, a.m. at ship in latitude 39° 53′ S., longitude about 11° 30′ E., with the following observations find the true and apparent altitudes of Mars, the error of the chronometer for Greenwich mean time, and the longitude of the place :—

Approx. M. T. G. by chro.	*Alt. moon's U. L.*	*Obs. dist. nearest limbs.*
Dec. 31 1h. 52m. 30s. a.m.	23° 10′ 0″	84° 28′ 20″
	I. E. + 2 16	I. E. + 3 14·3

Height of the eye 25 feet.

Ex. 533. 1887, March 17th, at 3h. 23m. 48s, a.m. mean time at place in latitude 33° 46′ N., longitude about 72° 21′ W., the distance between the nearest limb of the moon and Altair was observed to be 34° 17′ 20″, with a sextant whose index error was − 7′ 4″. If the chronometer at the instant showed 8h. 13m. 40s., find its error for mean time at Greenwich.

Ex. 534. 1887, January 6th, at 8h. 28m. 40s. p.m. mean time at ship in latitude 46° 35′ N., longitude about 43° 41′ W., the observed distance between the farthest limbs of the moon and Saturn was 41° 41′ 30″; index error + 2′ 55″. If the chronometer at the time showed 11h. 22m. 50s., find its error for Greenwich mean time.

Ex. 535. In the observation for longitude by a lunar distance distinguish between the "observed," the "apparent," and the "true distance." If the moon and a star have the same apparent altitude, explain (with figure) why the moon's true altitude will be greater than that of the star.

For Lieutenant, 1873.

Ex. 536. In the "Nautical Almanac" for July 1, 1877, the moon's distance from Jupiter is given as 76° 2′ 24″ at noon, and 77° 32′ 10″ at three o'clock. Explain clearly why the moon's distance from the sun and certain other bodies varies thus rapidly.

What do you mean by clearing a lunar distance? Why is the correction in altitude additive in the case of the moon, and subtractive in that of the sun, a planet, or a star?

A. 1877.

Ex. 537. In a lunar observation, the altitude of the sun being observed, show how the altitude of the moon may be deduced from it. *For Lieutenant,* 1873.

Ex. 538. For what purpose are "lunar distances" given in the "Nautical Almanac"? What corrections are needed to obtain the reduced from the observed distance?

Second B.Sc. London, 1879.

Ex. 539. Explain clearly, with well-drawn figures, the two principles on which the clearing of the lunar distance is made to depend. Give the investigation of the method derived from the second principle.

Ex. 540. Investigate the construction of the table of "auxiliary angle A," used in obtaining a true distance.

Royal Naval College, 1867.

Ex. 541. Clear the lunar distance from the effects of parallax and refraction by means of the versine method.

Royal Naval College, 1864.

Ex. 542. Explain clearly what are the elements of the two spherical triangles used in the process of clearing the lunar distance. State how the triangles are connected, and write down a formula which will enable you to find the correct distance.

A. 1882.

Ex. 543. Assuming the value of the cosine of an angle in a spherical triangle in terms of the sides, investigate an expression for clearing a lunar distance.

Royal Naval College, 1868.

Ex. 544. Describe the methods of finding the longitude at sea—(1) by chronometer; (2) by a lunar distance. Explain fully the principle of each of these methods. *A.* 1881.

Ex. 545. Explain the rule for finding the true distance of the moon (E. of meridian) from a star. Latitude 40° N.; apparent distance 58° 15′.

	R. A.	*Dec.*	*App. alt.*	*True alt.*
Moon	14 hrs.	20° N.	55° 10′	55° 40′ 0″
Star	10 hrs.	8 N.	45 0	44 59 5

Drawing the figure neatly on the plane of the equator.

Royal Naval College, 1868.

Ex. 546. The apparent and true altitudes of the moon are respectively 39° 9′ 2″ and 39° 50′ 14″, and those of the sun 16° 1′ 9″ and 15° 57′ 57″, and their apparent distance 88° 9′ 10″. If the true distance is approximately 87° 56′, find the true distance from the following expression:—

$$\left.\begin{array}{c}\text{Corr. of}\\ \text{dist.}\end{array}\right\} = \left\{\begin{array}{l}(+\tan. A\,.\,\cot. C - \sec. A\,.\,\sin. B\,.\,\text{cosec}. C)\\ \qquad\qquad \times \text{ correction of moon's altitude.}\\ (-\tan. B\,.\,\cot. C + \sec. B\,.\,\sin. A\,.\,\text{cosec}. C)\\ \qquad\qquad \times \text{ correction of sun's altitude,}\end{array}\right.$$

$$\left.\begin{array}{lr}\text{where moon's true altitude + moon's apparent altitude} & = 2A\\ \text{Sun's apparent altitude + sun's true altitude} & = 2B\\ \text{Approximate true distance + apparent distance} & = 2C\end{array}\right\}.$$

H. 1882.

Ex. 547. Find D from the formulæ:—

$$M = \sqrt{\cos. A_1\,.\,\sec. a\,.\,\cos. A\,.\,\sec. a_1\,.\,\cos. \tfrac{1}{2}(a + a_1 + d)\,.\,\cos. \tfrac{1}{2}(a + a_1 - d)}$$

$$\tan. \theta = \frac{M}{\sin. \frac{1}{2}(A + A_1)};\ \cos. \frac{D}{2} = \frac{M}{\sin. \theta}$$

when $A = 26°\ 17'\ 15''$ $a = 25°\ 28'\ 0''$ $d = 53°\ 37'\ 15''$
$A_1 = 70\ \ 2\ \ 0$ $a_1 = 70\ \ 2\ \ 30$

For Lieutenant, 1873.

Ex. 548. Investigate Airy's method of clearing lunar distances in which the mean between the apparent and corrected element is used. *H.* 1883.

Ex. 549. Find the distance between the moon and Regulus, having given R. A. of moon 16h. 12m. 19s., R. A. of Regulus, 10h. 1m. 14s. Declination of moon 16° 55′ 54″ N.; declination of Regulus 12° 37′ 15″ N. *For Lieutenant*, 1873.

CHAPTER XVIII.

Longitude by Jupiter's satellites—How found—Why not used when extreme accuracy is required—Precautions to be observed—Example — Exercise — Occultations — Definitions — Utility of the method—Limiting parallels—Proof of formulæ—Example—Exercise—Examination.

LONGITUDE BY JUPITER'S SATELLITES AND OCCULTATIONS.

GALILEO in 1610 discovered that Jupiter is attended by *four* satellites, the planes of whose orbits very nearly coincide with that of the equator of the planet, all being included within half a degree. The length of Jupiter's shadow (cast by the sun) is about 1240 semidiameters of the planet, whereas the distance of the most remote satellite is only twenty-six semidiameters; hence the satellites in their revolutions round their primary must pass behind the planet and be hid in his shadow, and also pass before the planet and be seen projected on his disc. The heliocentric distance of Jupiter is about 5·2 times that of the earth's, while the plane of his orbit makes an angle of only about $1\frac{1}{2}^{\circ}$ with that of the ecliptic; and therefore the same phenomena *almost* must be seen from the earth as from the sun. The only exception being that the *fourth* satellite, in extreme cases, is not eclipsed,

Jupiter's first satellite completes a revolution round the planet in forty-two terrestrial hours, or to an observer at the planet's centre it moves through a degree on the celestial concave every seven minutes. The second, third, and fourth satellites take 85, 170, and 400 hours respectively to complete a revolution. Thus the one which has the slowest motion travels nearly twice as fast among the fixed stars as our moon does.

APPEARANCES OF THE SATELLITES.—There are four different effects visible from the earth :—

(1) *The eclipse of the satellites,* or their passage through the shadow of the planet. Their entrance into the shadow is called an *immersion*, their exit an *emersion.*

(2) *The occultation of a satellite.*—This occurs when the planet itself hides the satellite from view.

(3) *The transit of a satellite over the planet* occurs when a satellite comes between the earth and the planet. The entrance of the satellite on the disc of Jupiter is called its *ingress*, and its leaving the disc is called its *egress*.

(4) *The transit of the shadow of a satellite.*—This corresponds to an eclipse of the sun with us. The shadow may be seen moving across the disc of the planet parallel to its belts.

The frequency with which some one or other of these phenomena can be seen would render this method for finding longitude peculiarly applicable if other circumstances did not prevent its use. They are all predicted in the "Nautical Almanac" for Greenwich mean time, and diagrams for north latitude for the inverting telescope are inserted for the middle of the month, except when in opposition. For south latitude these diagrams should be reversed. The mean time at ship should be found from the altitude of a celestial body near the prime vertical, and the error of a good watch ascertained before watching for any of the above events.

The following remarks express so clearly all that is necessary to be known on the subject for the practice of this method, that we have quoted them from the "Nautical Almanac" for 1870:—

"The eclipses of Jupiter's satellites, especially of the first (because of its quicker motion), afford us, perhaps, the readiest means of determining longitude; all that is necessary to be known being the exact time of observation. The difference between this time and the time at Greenwich shows the difference of longitude at once, and it is *east* or *west* of Greenwich, according as the time of observation is *greater* or *less* than the Greenwich time.

"Independent of defects in the tables, there are difficulties attending the observation of these phenomena which unfit them for *accurate* determinations of longitude. Different telescopes give different results; and care should be taken to have recourse to those corresponding observations which have been made under circumstances the most similar, and particularly with telescopes of the same quality and power. When extreme accuracy is not required, the eclipses of the satellites will always afford a good approximation towards the difference of meridians, and observations of them should on no account be neglected, especially when the disappearance and reappearance of the same satellite are both visible.

"Eclipses generally happen when the satellite is apparently at some distance from Jupiter, except near the opposition of Jupiter to the Sun, when the eclipse takes place near the planet. The disappearances and reappearances happen on the western side of the planet before opposition, but afterwards on the eastern side; with an inverting telescope the appearances will be the contrary. Before opposition the disappearances only of the first satellite are visible; and after the opposition the reapperances only. The disappearance and reappearance of the second satellite can seldom be observed at the same eclipse, but both phenomena are generally visible with the third and fourth satellites."

When practising this method of determining longitude, the magnifying power of the telescope employed should not be less than 40, and the Sun should not be more than 8° below the horizon, nor Jupiter less than 8° above it, for the phenomena to be distinctly visible. The times of eclipse are also modified by the clearness or otherwise of the atmosphere; and because the satellites have considerable apparent diameters as seen from Jupiter's centre, and the penumbra extends to a sensible, although a very small, distance beyond the shadow, the instant of total immersion or emersion cannot be exactly determined. Hence the only case which can be relied on is when the same person, with the same telescope, using the same power, on the same evening, under the same atmospheric and other conditions, observes both the immersion and emersion of the same satellite, and the mean of the longitudes thus found must then be taken as the correct one.

The following example is taken from the "Nautical Almanac" for 1870: "Suppose the disappearance of Jupiter's first satellite to be observed at Paris on July 24th, at 14h. 3m. 24·9s. mean time at place; by reference to page 466 it appears that the disappearance will take place at Greenwich at 13h. 54m. 4·3s. Greenwich mean time. The difference, 9m. 20·6s., is the difference of longitude between Greenwich and Paris; and because the Paris time is greater than that of Greenwich, we infer that Paris is to the east of Greenwich."

Exercise XXV.

Ex. 550. What do you mean by immersion, emersion, ingress, and egress when applied to Jupiter's satellites?

Ex. 551. What phenomena connected with Jupiter's satellites are visible from the earth?

Ex. 552. Explain generally the method by which the longitude may be obtained by observing the eclipses of Jupiter's satellites. *Honours*, 1872.

Ex. 553. Why should Jupiter's satellites not be used for accurate determination of longitude?

Ex. 554. State the methods of finding the longitude by chronometer, by eclipses of Jupiter's satellites or other celestial signals, and by electric telegraph.
Second B.Sc. London, 1873.

Ex. 555. Of what practical use in navigation are Jupiter s satellites? Explain how Roemar demonstrated the progressive transmission of light. *Second B.Sc. London*, 1879.

OCCULTATIONS.

DEFINITIONS.—The moon, being the nearest of all visible heavenly bodies, must necessarily pass between the earth and every other celestial object in her path on the celestial concave; and her diameter being about half a degree, she must in her course hide any object which is not more than 15′ from the path of her centre.

When a star or planet is hid from view by the interposition of the moon, the object is said to be *occulted;* the phenomena is called an *occultation*, and is one of the most remarkable in astronomy. The disappearance always takes place on the eastern limb of the moon, because her motion is from west to east. From new moon to full the dark limb is foremost, hence the object will disappear at the dark and reappear at the enlightened edge; and the opposite of this is witnessed between full and new moon. The disappearance is called an *immersion*, and the reappearance an *emersion.*

UTILITY.—"Like all other astronomical events which can be accurately predicted, and whose occurrence is instantaneous, this phenomena may be used for the determination of terrestrial longitude. The utility of this method is increased by its happening so frequently; and it may be reckoned among the best means which science has supplied for finding longitude."

GENERAL EXPLANATION.—Because the fixed stars are so distant, all straight lines drawn from a star to any two places on the earth's surface will be parallel; and hence we may conceive these lines to be situated in the surface of a cylinder (whose axis is the line joining the centre of the moon and the star) enveloping the moon and extending to the earth. If an occultation of a star be observed, the point of observation must be some

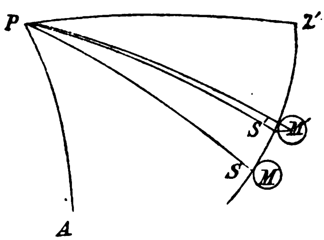
place on the earth's surface enclosed by the cylinder, and therefore all observers on that curve will see the occultation at the same instant of absolute time. The event is instantaneous, and can be observed with a telescope alone—that is, without the aid of such instruments as are liable to derangement, and therefore the method is susceptible of the strictest accuracy.

The right ascension of the star occulted will be the apparent right ascension of the point of the limb of the moon at which the occultation takes place; and if by any process we can remove the effects of parallax on the moon, and thus ascertain the right ascension of her centre at the same moment, we can, by interpolation between the times in the "Nautical Almanac," find the exact Greenwich date at which the event occurred. If the time by chronometer be noticed at the instant of immersion or emersion, its error is then deduced.

Limiting Parallels, and Proof.—"The parallels of latitude beyond which the star cannot be occulted by the moon," called *limiting parallels*, are recorded in the "Nautical Almanac," and from these entries an observer can see if the phenomena will occur at the place where he is situated.

Let P be the elevated pole, Z' the reduced zenith, therefore $P\,Z'$ is the reduced colatitude of the observer, $P\,A$ the hour circle through the first point of Aries, S the place of the star, and $Z'\,P\,S$ its hour angle at the occultation, M the apparent place of the moon corrected for refraction, M' her true place, S' the true place of the point of the moon's limb at which the occultation took place. To find the longitude, the mean time at place must be accurately known. and the Greenwich time approximately. The former is ascertained beforehand by the altitude of an object near the prime vertical; and the latter by applying the longitude by account in time to the mean time at ship, or may be deduced from the chronometer.

Fig 61.



Then

$Z'PS$ the star's hr. $\angle$ = mean sun's R. A. + M.T. place − star's R. A.

SS' the moon's par. in alt. = moon's hor. par. × sin. app. zen. dist.

$= H\,.\,\text{sin.}\ Z'\,S$ very nearly.

If H be the moon's horizontal parallax, $P\,S$ is the star's polar

distance, $P\ M'$ the moon's polar distance, $A\ P\ S$ the star's right ascension, $A\ P\ M'$ the moon's right ascension, which is required to be found.

In the figure it is seen

$$A\ P\ M' = A\ P\ S + S\ P\ S' + S'\ P\ M'$$

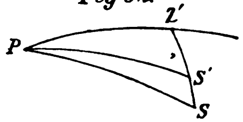
and $S\ P\ S'$ is called the *parallax in right ascension*, and $S'\ P\ M'$ the *semidiameter in right ascension.*

Several methods have been proposed by different authors for finding these quantities, but the following being the direct solution of two spherical triangles, and not involving any constant numbers, appears to us the simplest and easiest to be remembered.

The investigation being of some length, distinct steps are proposed as independent problems.

To find the parallax in R. A. and in P. D.

Fig 62.



Let fig. 62 represent similar quantities taken from fig. 61, in which are known $P\ Z'$ the colatitude, $P\ S'$ the star's polar distance, and $Z'\ P\ S$ the star's hour angle.

By Napier's "Analogies"—

$$\left.\begin{array}{l}(1)\ \tan.\ \tfrac{1}{2}\,(Z'+S) = \cos.\ \tfrac{1}{2}\,(P\,S \backsim P\,Z').\ \sec.\ \tfrac{1}{2}\,(P\,S+P\,Z').\ \cot.\ \tfrac{1}{2}\,Z'\,P\,S \\ (2)\ \tan.\ \tfrac{1}{2}\,(Z' \backsim S) = \sin.\ \tfrac{1}{2}\,(P\,S \backsim P\,Z').\ \mathrm{cosec.}\ \tfrac{1}{2}\,(P\,S+P\,Z').\ \cot.\ \tfrac{1}{2}\,Z'P\,S\end{array}\right\}$$

The student should bear in mind that the greater side is subtended by the greater angle; and that $\frac{1}{2}\ (P\ S + P\ Z')$ and $\frac{1}{2}\ (Z' + S)$ are of the same affection.

Hence the angles Z' and S are known.

The zenith distance of the star may be calculated from the formula—

$$\sin.\ Z'\ S = \sin.\ P\,Z'\ \frac{\sin.\ Z'\ P\,S}{\sin.\ S};$$

and because the zenith distance must always be less than 90°, no ambiguity can exist. Hence $Z'\ S$ is known,

and $\quad S\,S' = H\ .\ \sin.\ Z'\ S$ very nearly,

then $\quad Z'\,S' = Z'\,S - S\,S'$.

Therefore in the spherical triangle $Z'\ P\ S'$ are now known, $P\ Z'$, $Z'\ S'$, and the included angle $P\,Z'\ S'$, and by a precisely similar method to the last the angle $Z'\ P\ S'$ and $P'\ S'$ may be found. In rare instances, viz. when near 90°, there may be an uncertainty about $P\ S'$ obtained by this method. It

should then be calculated by some other method, as one of Gauss's Theorems.

Now $S\,P\,S'$ or parallax in R. A. $= Z'\,P\,S - Z'\,P\,S'$
$= \text{star's hr.} \angle - Z'\,P\,S$ . . . (α)

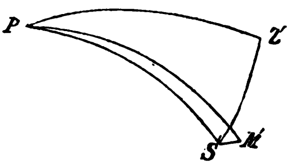
$S\,P\,S'$, reckoned in time, is subtractive when the star is east of the meridian, and additive when it is west.

And parallax in $P\,D = P\,S - P\,S'$
$= \text{star's } P\,D - P\,S'$ . . . (β)

To find the semidiameter in R. A.

The letters in fig. 63 correspond to those in fig. 61, from which they are taken.

Fig 63.



In the spherical triangle $P\,S'\,M'$ are known, $P\,S'$ from the last problem, $P\,M'$ the moon's polar distance, and $S'\,M'$ the moon's geocentric semidiameter, and therefore the same as taken from the "Nautical Almanac," and it is required to find the angle $M'\,P\,S'$.

By a well-known formula in spherical trigonometry

$$\sin.^2\tfrac{1}{2}\,S'PM' = \text{cosec.}PS'.\,\text{cosec.}PM'.\,\sin.\tfrac{1}{2}\{M'S' + (PM' \backsim PS')\}$$
$$\times \sin.\ \tfrac{1}{2}\ \{M'\,S' - (P\,M' \backsim P\,S')\}.$$

But the quantities $S'\,P\,M'$, $\{M'\,S' + (P\,M' \backsim P\,S')\}$, and $\{M'\,S' - (P\,M \backsim P\,S')\}$ are all so small that the number of seconds in each multiplied by sin. 1″ may be written for the sines of the quantities.

$$\therefore \left(\frac{S'PM'}{2}\right)^2.\ \sin.^2\,1'' = \text{cosec. } PS'.\ \text{cosec. } PM'.$$
$$\times \frac{M'S' + (PM' \backsim PS')}{2}.\ \sin.\ 1'' \times \frac{M'\,S' - (P\,M' \backsim P\,S')}{2}.\ \sin.\ 1''.$$

Dividing by $\dfrac{\sin.^2\,1''}{4}$, and reducing arc to time, we get $(S'\,P\,M')^2$

$$\text{seconds} = (\tfrac{1}{15})^2.\ \text{cosec. } P\,S'.\,\text{cosec.}P\,M'\{M'\,S' + (P\,M' \backsim P\,S')\} \times$$
$$\{M'\,S' - (P\,M' \backsim P\,S')\}.$$

Hence $S'\,P\,M'$, or semidiameter in R. A., is known, and is subtractive for an immersion, but additive for an emersion.

Having found the parallax in R. A. and semidiameter in R. A., we can now obtain the R. A. of the moon's centre.

For in fig. 61 (emersion W. of the meridian)

z

$$A\,P\,M' = A\,P\,S + S\,P\,S' + S'\,P\,M',$$

or R. A. of moon's centre = R. A. of star + parallax in R. A. + semidiameter in R. A.

Care must be taken to mark the quantities we have found with the proper signs.

When the right ascension of the moon's centre is found, take her right ascension for the two hours between which the true right ascension lies; then by simple proportion the Greenwich date corresponding to the occultation can be found, and if it differ considerably from that by account, the semidiameter in R. A. must be recomputed, because the declination of the moon should be known with accuracy to obtain a true result.

The following example is taken from the "Monthly Notices of the Royal Astronomical Society" for December, 1866 :—

Ex. 556. 1866, November 22nd, at 10h. 41m. 4·8s. mean time at Waterloo, near Liverpool, in latitude 53° 28′ 24″ N., and longitude in time 12m. 6·9s. W., the emersion of Aldebaran was observed by John Joynson, Esq. Find the true Greenwich date of observation.

For Greenwich mean time.

M. T. at place,	Nov.	22d.	10h.	41m.	4·8s.
Long. in time			+	12	6·9
G. M. T.,	Nov.	22	10	53	11·7

For moon's semidiameter.

Nov. 22, 10·9h.	16′ 46·7″
	60
	1006·7

For horizontal parallax.

Nov. 22, 10·9h.	61′ 28·1″
Reduction	− 7·8
True H. P.	61 20·3
	60
H. =	3680·3

For moon's polar distance.

Declination at 10h.	16° 58′ 36·8″ N.
Cor. for 53m. 12s.	+ 4 13·9
True declination	17 2 50·7 N.
N. P. D.	72 57 9·3

For reduced colatitude.

Latitude	53° 28′ 24″ N.
Reduction	11 1
Reduced lat.	53 27 23 N.
Reduced colat.	36 42 37

For star's R. A. and P. D.

Star's R. A.	4h. 28m. 18·49s.
,, declination	16° 14′ 14·9″ N.
,, N. P. D.	73 45 45·1

For star's hour angle.

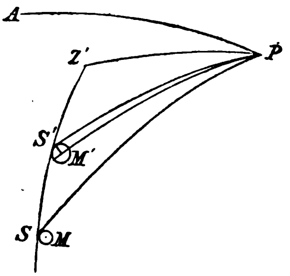
M. T. at place	10h.	41m.	4·8s.
R. A. mean sun	16	6	45·7
R. A. meridian	26	47	50·5
R. A. Aldebaran	4	28	18·5
Star's W. hr. ∠	22	19	32
„ E. hr. ∠	1	40	28
In arc	25°	7′	0″
½ polar angle	12	33	30 and star is E. of meridian.

For R. A. of mean sun.

At noon, Nov. 22		16h.	4m.	58·37s.
Acceleration for	10h.		1	38·56
	53m.			8·71
	11·7s.			·03
R. A. mean sun		16	6	45·67

The adjoining figure illustrates the present question, viz. an emersion E. of the meridian.

Fig 64



To solve the triangle Z′ P S.

$Z' P$	36° 42′ 37″	Let polar ∠ = P
$P S$	73 45 45	
$P S + Z' P$	110 28 22	∴ ½ $(P S + Z' P)$ 55° 14′ 11″
$P S - Z' P$	37 3 8	∴ ½ $(P S - Z' P)$ 18 31 34

½ $(P S - Z' P)$	18° 31′ 34″	cos.	9·976890	sin.	9·502068
½ $(P S + Z' P)$	55 14 11	sec.	·243979	cosec.	·085386
½ polar ∠	12 33 30	cot.	·652157	cot.	·652157
½ $(Z' + S)$	82 22 12·5	tan.	10·873026		10·239611
½ $(Z' - S)$	60 3 36	tan.			

∴ Z' = 142 25 48·5 and S = 22° 28′ 36·5″

For the side Z′ S.

Sin. Z′ S = sin. Z′ P . sin. Z′ P S . cosec. S.

$Z' P$	36° 42′ 37″	sin.	9·776533
$Z' P S$	25 7 0	sin.	9·627840
S	22 18 36·5	cosec.	·420651
$Z' S$	41 56 32	sin.	9·825024

To find parallax in alt.

H	3680·3″	log.	3·565883
$Z D$	41° 56′ 32″	sin.	9·825024
$S S'$	2460″	log.	3·390907

∴ Parallax in alt. 41′ 0″

To find Z′ S′.

$Z' S$	41° 56′ 32″
$S' S$	— 41 0
$Z' S'$	41 15 32

To solve the triangle P Z' S'.

PZ'	36° 42′ 37″	$\frac{1}{2}Z' = 71° 12' 54''$
$Z'S'$	41 15 32	
$Z'S' + PZ'$	77 58 9	$\therefore \frac{1}{2}(Z'S' + PZ') = 38\ 59\ 4{\cdot}5$
$Z'S' - PZ'$	4 32 55	$\therefore \frac{1}{2}(Z'S' - PZ') = 2\ 16\ 27{\cdot}5$

Let $P' = Z'PS'$ and $S' = PS'Z'$.

$\frac{1}{2}(Z'S' - PZ')$	2° 16′ 27·5″	cos.	9·999658	. .	sin.	8·598609
$\frac{1}{2}(Z'S' + PZ')$	38 59 4·5	sec.	·109403	. .	cosec.	·201272
$\frac{1}{2}Z'$	71 12 54	cot.	9·531652	. .	cot.	9·531652
$\frac{1}{2}(P' + S')$	23 36 59·5	tan.	9·640713			8·331533
$\frac{1}{2}(P' - S')$	1 13 44·9					

P' =	24 50 44·4	and	$S' = 22° 23' 14{\cdot}6''$
P =	25 7 0		
∴ Par. in R. A.	16 15·6		

In time = − 1m. 5·04s. This is − because star is E. of meridian.

To find the side P S'

$$\sin. PS' = \sin. Z'P \,.\, \sin. Z' \,.\, \operatorname{cosec.} S'.$$

$Z'P$	36° 42′ 37″	sin.	9·776533
Z'	142 25 48·5	sin.	9·785136
S'	22 23 14·6	cosec.	·419226
PS'	73 7 45·3	sin.	9·980895
PS	73 45 45·1		
Par. in polar dist.	37 59·8		

To find the angle S' P M' or semidiameter in R. A.

PS'	73° 7′ 45·3″	
PM'	72 57 9·3	
$PS' - PM'$	10 36	$S'M' + (PS' - PM') = 27' 22{\cdot}7'' = 1642{\cdot}7''$
$S'M'$	16 46·7	$S'M' - (PS' - PM') = 6\ 10{\cdot}7 = 370{\cdot}7$

$$(S'PM')^2 = \frac{1}{15^2}\,.\operatorname{cosec.} PS'\,.\operatorname{cosec.} PM'\,.\left\{S'M' + (PS' - PM')\right\}.\left\{S'M' - (PS' - PM')\right\}$$

PS'	73° 7′ 45·3″	cosec.	·019105
PM'	72 57 9·3	cosec.	·019514
$S'M' + (PS' - PM')$	1642·7	log.	3·215558
$S'M' - (PS' - PM')$	370·7	log.	2·569023
			5·823200
	Log. 15 × 2		2·352182
			2)3·471018

∴ $S'PM'$ or semidiameter in R. A. = + 54·39s. log. 1·735509

This is + because it is an emersion.

To find the R. A. of moon's centre and Greenwich date.

R. A. moon's centre = R. A. of star − par. in R. A. + semidia. in R. A.

R. A. Aldebaran	4h.	28m.	18·49s.				
Parallax in R. A.		− 1	5·04				
R.A. true point of contact	4	27	13·45	R.A. moon at 10h.	4h.	25m.	46·65s.
Semidiameter in R. A.		+	54·39	„ 11	4	28	25·62
True R.A. moon's centre	4	28	7·84	Diff. in R. A. for 1h.		2	38·97
Nov. 22 R. A. at 10h.	4	25	46·65				
Diff. in R. A. for G. M. T.		2	21·19				

We have now a difference in R. A. of the moon of 2m. 38·97s. for one hour to find what interval will give a difference in R. A. of 2m. 21·19s.; and as the motion of the moon in R. A. for this date is so constant, the interval required may be found by simple proportion, and gives 53m. 17·3s.

Hence G. M. T. of occultation was 10h. 53m. 17·3s., thus showing that the assumed G. M. T. was correct within 5·6s.

Exercise XXVI.

Ex. 557. To what other celestial phenomena are occultations allied?

Ex. 558. What renders the calculation for an occultation easier than those for an eclipse?

Ex. 559. What is the meaning of the terms occultation, immersion, and emersion? Mention briefly the object of, and the principal corrections in, the observation of an occultation.

Ex. 560. On which limb of the moon does an emersion take place?

Ex. 561. Explain the rule for finding the longitude by an occultation. *For Beaufort Testimonial*, 1865.

Ex. 562. In the problem of finding the longitude by an occultation of a star by the moon, describe briefly the different steps in calculating the right ascension of the true place of the moon's centre. Show at length how to find the semidiameter in R. A. *Honours*, 1873.

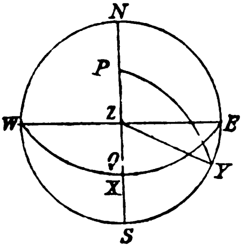
Exercise XXVII.

We shall now work in full a few examples as a guide to the student; and then several others will be added to test his power of dealing with such questions.

MISCELLANEOUS EXAMPLES.

Ex. 563. In latitude 45° N. the meridian altitude of the sun was 30°; show that the tangent of quarter the length of the day was $\frac{1}{\sqrt[4]{3}}$.

Fig 65



Let $N\ E\ S\ W$ be a projection of the celestial sphere on the plane of the horizon; Y the place of the sun at rising, and X his place on the meridian.

Then $Z\ Y$ is 90°, or a quadrant, and in the quadrantal triangle $P\ Z\ Y$ are given $P\ Z = 45°$

$$P\ Y = P\ X = P\ Q + Q\ X = 90° + 15° = 105°,$$
$$\text{and } Z\ Y = 90°.$$

Let the angle $Z\ P\ Y = h$;

$$\text{then sin.}\left(\frac{\pi}{2} - h\right) = -\text{cot. } P\ Y\ .\ \text{cot. } P\ Z$$

$$\text{cos. } h = -\text{cot. } 105°\ .\ \text{cot. } 45°$$
$$= \text{tan. } 15°.$$

$$\frac{1 - \text{cos. } h}{1 + \text{cos. } h} = \frac{1 - \text{tan. } 15°}{1 + \text{tan. } 15°}$$

$$\frac{2\ \text{sin.}^2 \frac{h}{2}}{2\ \text{cos.}^2 \frac{h}{2}} = \frac{\text{tan. } 45° - \text{tan. } 15°}{1 + \text{tan. } 45°.\ \text{tan. } 15°}$$

$$\text{tan.}^2 \frac{h}{2} = \text{tan. } (45 - 15)°$$
$$= \text{tan. } 30°.$$

But h is half the length of the day, and tan. $30° = \frac{1}{\sqrt{3}}$;

$\therefore$ tan. $\frac{1}{4}$ length of day $= \frac{1}{\sqrt[4]{3}}$. Q. E. D.

Ex. 564. If a be the altitude of the sun on the prime vertical, and a' its altitude on the six o'clock hour circle, find expressions for determining his declination, and the latitude of the place of observation.

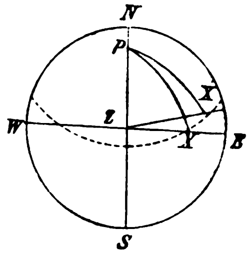
Let X' be the sun on the six o'clock hour circle,

X „ „ prime vertical;

then $ZX' = (90° - a')$ and $ZX = 90° - a$.

If d be the declination, $P\ X'$ or $P\ X = 90° - d$, and l be the latitude, $P\ Z = 90° - l$.

Fig. 66



In the triangle $X'P\ Z$, the angle $X'P\ Z$ is six hours, or a right angle;

$\therefore$ cos. $Z\ X' =$ cos. $P\ X'$. cos. $P\ Z$.

Hence cos. $P\ Z = \dfrac{\text{cos. } Z\ X'}{\text{cos. } P\ X'}$. I.

In the triangle $P\ Z\ X$ the angle $P\ Z\ X$ is a right angle, and

$$\text{cos. } P\ X = \text{cos. } P\ Z\ .\ \text{cos. } Z\ X;$$

$$\therefore \text{cos. } P\ Z = \frac{\text{cos. } P\ X}{\text{cos. } Z\ X} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{II.}$$

Equating I. and II.—

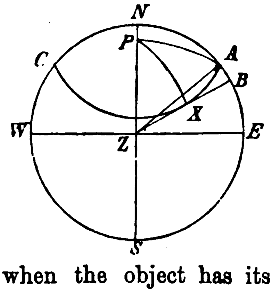
$$\frac{\text{cos. } Z\ X'}{\text{cos. } P\ X'} = \frac{\text{cos. } P\ X}{\text{cos. } Z\ X} \text{ or } \frac{\text{sin. } a'}{\text{sin. } d} = \frac{\text{sin. } d}{\text{sin. } a};$$

$$\left.\begin{array}{l} \text{i.e. sin.}^2\ d = \text{sin. } a'.\ \text{sin. } a\ ;\ \therefore \text{sin. } d = \sqrt{\text{sin. } a'.\ \text{sin. } a} \\ \text{From I. sin. } l = \dfrac{\text{sin. } a'}{\text{sin. } d} = \dfrac{\text{sin. } a'}{\sqrt{\text{sin. } a'.\ \text{sin. } a}} = \sqrt{\dfrac{\text{sin. } a'}{\text{sin. } a}} \end{array}\right\} \textit{Answer.}$$

Ex. 565. If the declination of an object be greater than the latitude of the place, both being of the same name, find the greatest azimuth of the body, the time when it will have its greatest azimuth, and its altitude at that time, that is, when it will appear to move vertically.

Let $N\ E\ S\ W$ be the plane of the horizon, $N\ S$ the meridian, $E\ Z\ W$ the prime vertical, P the elevated pole, $A\ X\ C$ the parallel of declination on which the object moves, X the position of the object when it has the azimuth $P\ Z\ X$, $P\ Z\ A$ its azimuth at rising.

Fig 67



Then $N\ Z\ B - N\ Z\ A = A\ Z\ B$ will be the number of degrees which the object will appear to increase in azimuth.

$Z\ P\ X$ will be the hour angle when the object has its maximum azimuth.

$Z\ X$ will be the zenith distance when the object has its maximum azimuth.

Now the angle which $P X$ makes with $Z B$ is a right angle, hence we have given in the triangle $P Z X$ the polar distance $P X$, the colatitude $P Z$, and the right angle $P X Z$ to find—

I. The greatest azimuth $P Z X$.
II. The hour angle $Z P X$ at the time of max. az.
III. The altitude $B X$ at the time of max. az.

And $\quad \sin. P X = \sin. P Z . \sin. P Z X;$

$$\therefore \sin. P Z X = \sin. P X . \operatorname{cosec}. P Z,$$

$$\text{or } \sin. \text{az.} = \cos. \text{dec.} . \sec. \text{lat.} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{I.}$$

Again—

$$\cos. Z P X = \tan. P X . \cot. P Z,$$

$$\text{or } \cos. h = \cot. \text{dec.} . \tan. \text{lat.} \quad . \quad . \quad . \quad . \quad . \quad \text{II.}$$

Once more—

$$\cos. P Z = \cos. P X . \cos. Z X;$$

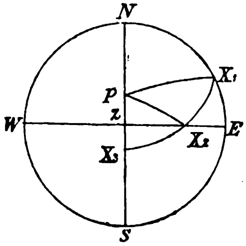
$$\therefore \cos. Z X = \cos. P Z . \sec. P X$$

$$\sin. B X = \sin. N P . \sec. P X,$$

$$\text{or } \sin. \text{alt.} = \sin. \text{lat.} . \operatorname{cosec}. \text{dec.} \quad . \quad . \quad . \quad . \quad . \quad \text{III.}$$

Ex. 566. Given the sun's declination, and that he is due E. when half the time between his rising and noon has elapsed. Find the latitude of the place.

Fig 68.



Let X_1, X_2, and X_3 be the positions of the sun at rising, when on the prime vertical, and when on the meridian. Then by hypothesis, $X_1 P X_2 = X_2 P X_3$ each $= \frac{h}{2}$ if h be the hour angle at rising.

From the fundamental formula

$$\cos. h = \frac{\cos. z - \cos. l'. \cos. p}{\sin. l'. \sin. p}$$

at rising $z = 90°$ $\quad = - \tan. l . \tan. d;$

$$\therefore 2 \cos.^2 \frac{h}{2} = 1 - \tan. l . \tan. d$$

$$\cos. \frac{h}{2} = \{\tfrac{1}{2} (1 - \tan. l . \tan. d)\}^{\frac{1}{2}} \quad . \quad . \quad \text{I.}$$

In the right-angled triangle $P Z X_2$, right angled at Z,

$$\cos. Z P X_2 = \tan. P Z . \cot. P X_2$$

$$\text{i.e. } \cos. \frac{h}{2} = \cot. l . \tan. d . \quad . \quad . \quad . \quad . \quad \text{II.}$$

Equating I. and III., and squaring—

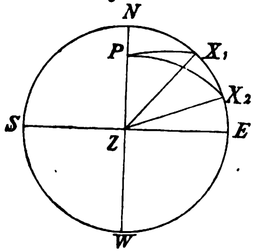
$$\tfrac{1}{2}(1 - \tan.\ l \,.\, \tan.\ d) = \cot.^2 l \,.\, \tan.^2 d$$
$$1 - \tan.\ l \,.\, \tan.\ d = 2 \cot.^2 l \,.\, \tan.^2 d;$$
$$\therefore \tan.^3 l - \tan.^2 l \,.\, \cot.\ d = -2 \tan.\ d.$$

From which cubic equation l can be found if d be known.

Ex. 567. Two stars whose R. A. and declinations are known were observed to rise at the same moment. Required the latitude of the place of observation.

Let X_1 and X_2 be the positions of the stars at rising, p_1 and h_1, elements of X_1; p_2 and h_2 elements of X_2; l the latitude.

Fig 69.



The general equation is

$$\cos.\ h = \frac{\cos.\ z - \cos.\ p \,.\, \cos.\ l'}{\sin.\ p \,.\, \sin.\ l'}$$

at rising $= - \cot.\ p \,.\, \cot\ l'$

$= - \tan.\ \text{dec.}\ \tan.\ \text{lat.}$

Hence, $\cos.\ h_1 = - \tan.\ d_1 \,.\, \tan.\ l,$

and $\cos.\ h_2 = - \tan.\ d_2 \,.\, \tan.\ l$

$$\frac{\cos.\ h_1}{\cos.\ h_2} = \frac{\tan.\ d_1}{\tan.\ d_2}$$

$$\frac{\cos.\ h_1 + \cos.\ h_2}{\cos.\ h_1 - \cos.\ h_2} = \frac{\tan.\ d_1 + \tan.\ d_2}{\tan.\ d_1 - \tan.\ d_2}$$

Reducing $$\frac{2\cos.\frac{h_1+h_2}{2} \,.\, \cos.\frac{h_1-h_2}{2}}{-2\sin.\frac{h_1+h_2}{2} \,.\, \sin.\frac{h_1-h_2}{2}} = \frac{\sin.\ (d_1+d_2)}{\sin.\ (d_1-d_2)};$$

$$\therefore -\cot.\frac{h_1+h_2}{2} = \sin.\ (d_1+d_2) \,.\, \text{cosec.}\ (d_1-d_2) \,.\, \tan.\frac{h_1-h_2}{2}$$

Now $\frac{h_1-h_2}{2}$ is the difference of R. A., and is therefore known, and hence $\frac{h_1+h_2}{2}$ is easily deduced from the formula, whence h_1 and h_2 can be found.

Then in the triangle $Z P X_1$ are known p_1, h_1, and $Z X_1$, a quadrant;

$$\therefore \sin.\ h_1 = \cot.\ p_1 \,.\, \cot.\ l'.$$

and tan. lat. $= \sin.\ h_1 \,.\, \cot.$ dec. . . Q. E. D.

This question may also be solved thus :—

In the triangle $P X_1 X_2$ we know p_1, p_2 and the angle $X_1 P X_2$, which is the difference of R. A., hence find $P X_1 X_2$, then $P X_1 N = 180° - P X_1 X_2$.

In the triangle $P N X_1$ are then known p_1, the angle $P X_1 N$, and the right angle $P N X$;

$$\therefore \text{ sin. } P N = \text{sin. } p_1 \text{ . sin. } P N X.$$

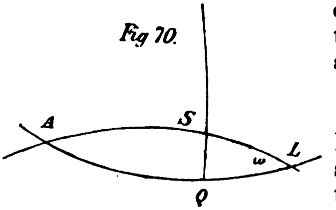
Ex. 568. If l be the latitude of a place at which, a month before the autumnal equinox, the day is as long as the longest day at a place in latitude l', show that

$$\text{tan. } l = \text{tan. } l' \sqrt{1 + 3 \text{ sec.}^2 \omega} \text{ approximately,}$$

ω being the obliquity of the ecliptic.

Let A be the first point of Aries, and L the first point of Libra, S the position of the sun in the ecliptic $A S L$, $A Q L$ a portion of the equinoctial, and $Q S$ a portion of a declination circle through S, the position of the sun one month before the autumnal equinox;

Fig 70.



$$\therefore S L = \frac{1}{12} \text{ of the ecliptic} = \frac{\pi}{6}$$

In triangle $S L Q$:—

sin. $S Q$ = sin. $S L$. sin. $S L Q$; that is—

$$\text{sin. dec.} = \text{sin. } \frac{\pi}{6} \text{ . sin. } \omega = \tfrac{1}{2} \text{ sin. } \omega \quad . \quad . \quad . \quad . \quad \text{I.}$$

Now for length of day in latitude l when sun has declination d,

$$\text{cos. } h = - \text{tan. } d \text{ . tan. } l \quad . \quad . \quad . \quad . \quad . \quad \text{II.}$$

the longest day in latitude l' must be when the sun has the greatest declination, and then $d' = \omega$

$$\text{cos. } h' = - \text{tan. } d' \text{ . tan. } l'$$
$$= - \text{tan. } \omega \text{ . tan. } l'.$$

And as II. and III. are equal by hypothesis—

$$- \text{tan. } d \text{ . tan. } l = - \text{tan. } \omega \text{ . tan. } l' \quad . \quad . \quad \text{III.}$$
$$\therefore \text{ tan. } l = \text{tan. } l' \text{ . tan. } \omega \text{ . cotan. } d$$

from I. $$= \text{tan. } l' \left\{ \frac{\text{sin. } \omega}{\text{cos. } \omega} \cdot \frac{\sqrt{4 - \text{sin.}^2 \omega}}{\text{sin. } \omega} \right\}$$
$$= \text{tan. } l' \text{ . } \sqrt{3 \text{ sec.}^2 \omega + 1} \quad . \quad \text{Q. E. D.}$$

MISCELLANEOUS QUESTIONS.

Ex. 569. Define :—Celestial meridian, the ecliptic, the first point of Aries, the horizon, and R. A. Explain by diagrams. What do you understand by the angular distance between two *objects*? What is the angular distance between two stars which

are both on the prime vertical, the altitude of one and the zenith distance of the other being 22° 30′? *E.* 1880.

Ex. 570. Show by a figure that a person elevated above the surface of the earth should see less of the earth and more of the heavens than a hemisphere; and explain the difference between the sensible and rational horizons. Define the latitude and longitude of a heavenly body, and show the difference between the longitude and right ascension of an object. *A.* 1873.

Ex. 571. Define the terms equator and ecliptic, and explain clearly the apparent motion of the sun in the latter.

B.A London, 1841.

Ex. 572. Define latitude and longitude of a place on the earth's surface, and also of a star. Define the right ascension and declination of a star. Distinguish between a great circle and a small circle on a sphere. Give some instances of each. Why is the time of one complete revolution of the earth on its axis not equal to a mean solar day? Determine the ratio of these two periods, assuming 365 days to be exactly one year.

B.A. and B.Sc. London, 1860-1867.

Ex. 573. Explain by means of figures the following corrections in Nautical Astronomy:—(*a*) Refraction. (*b*) Parallax. (*c*) Dip. (*d*) Augmentation of the moon's semidiameter. (*e*) The contraction of the moon's semidiameter. Investigate formulæ for computing the last two corrections.

Royal Naval College, 1866.

Ex. 574. Given the R. A. of the sun and the obliquity of the ecliptic, investigate an expression for his longitude.

Royal Naval College, 1866.

Ex. 575. The sun's longitude = 164° 33′, the obliquity of the ecliptic = 23° 27′ 45″. Calculate the R. A. and the declination.

Ex. 576. When is the sun's declination equal to the obliquity of the ecliptic? Prove by rules of spherical trigonometry.

Royal Naval College, 1867.

Ex. 577. Given the sun's longitude 198°, the moon's latitude 5°, and longitude 100°, to find the distance of the sun from the moon. *Royal Naval College,* 1869.

Ex. 578. Given the sun's R. A. = 4h., and the obliquity of the ecliptic = 23° 27′ 45″, to find his declination and longitude. *Royal Naval College,* 1868.

Ex. 579. Required the distance between the stars whose declinations are 30° N. and 20° S. respectively, and difference of R. A. 1h. 32m. 30s. *Royal Naval College,* 1865.

Ex. 580. The latitude of the moon is 3° 42′ 30″ N., and the difference of longitude between the sun and moon is 136° 32′. Required their distance.

Royal Naval College, 1865.

Ex. 581. Show how the distance and magnitude of the moon may be determined from the following data:—Moon's semidiameter 15′ 22′, and horizontal parallax 55′ 35″.

Royal Naval College, 1865.

Ex. 582. Supposing the moon's orbit to be an ellipse, and in the month of October, 1887, her greatest and least semidiameters to be 16′ 45·8″ and 14′ 43·5″, find the eccentricity of her orbit.

Ex. 583. Explain what you mean by the index error of a sextant, and describe the methods of determining it. How do you proceed to find whether the horizon glass is in adjustment?

E. 1874.

Ex. 584. The arc of a sextant is graduated to twice the number of degrees due to its length; show by this means the arc read off is equal to the angular distance between the two objects observed. *Royal Naval College*, 1866.

Ex. 585. Construct a figure and show what is meant by *sidereal time*, *apparent solar time*, *mean solar time*, and *equation of time.* *A.* 1868.

Ex. 586. Show how apparent time can be converted into mean time. The apparent time at a given place is 2h. 25m. 18s., and the mean time is 2h. 28m. 22s.; what is the equation of time? Is it additive to or subtractive from apparent time? There are two places, A and B; the longitude of A is 35° 18′ E., and when it is 2h. 30m. p.m. at A, it is 0h. 18m. a.m. at B. What is the longitude of B? *E.* 1872.

Ex. 587. Explain clearly the difference between mean solar and sidereal time, and investigate a formula for converting an interval of time expressed in sidereal time into mean solar time. *Example:* Convert 7h. 40m. of sidereal time into mean solar time. *Honours*, 1874.

Ex. 588. Define sidereal time, and state clearly what it is that is given under this heading in the "Nautical Almanac," p. II.

Given mean time 17h. 50m. 10s., and the right ascension of the mean sun 7h. 22m. 59s., find the sidereal time. Draw a figure for this example on the plane of the celestial equator, showing the relative positions of the meridian, the mean sun, and the first point of Aries. *A.* 1879.

Ex. 589. Define :—Sidereal day, mean solar day. Which of these two is the greater, and by how much? Explain clearly the reason of your answer. By help of your "Nautical Almanac" describe what are the relative positions of the mean sun and the true sun on 13th and 14th June. *A.* 1880.

Ex. 590. Why do the fixed stars appear to circulate from E. to W., and the sun to travel in the ecliptic from W. to E.? State and account for the directions of the real motions to which those apparent ones are due.

B.A. and B.Sc. London, 1871.

Ex. 591. Given sidereal time = 0h. 32m. 10s., and the R. A. of the mean sun at mean noon at place = 9h. 42m. 14·5s. Required the correct mean time.

Ex. 592. The H. A. of a star was 19h. 7m. 10s., R. A. of the star 6h. 50m. 9s., declination 30° N., R. A. of the mean sun at the preceding Greenwich mean noon 12h. 4m. 29s. Draw a figure on the plane of the equator, and find exact ship M. T., longitude being 60° E. *Royal Naval College*, 1873.

Ex. 593. Draw a diagram on the plane of the equator showing sidereal time, mean solar time, and equation of time; the sun's R. A. being 3h., sun's declination being 18° 18′ S., sun's hour angle = 2h., equation of time 13m. additive to apparent time. *Royal Naval College*, 1868.

Ex. 594. In a mean solar day a meridian of the earth revolves through 360° 59′ 8·33″. Explain this.

Royal Naval College, 1864.

Ex. 595. What is the cause of the error of parallax in taking an observation of the altitude of sun, moon, or planet? When is the correction for refraction the greatest, and when least? Explain this. Explain clearly why the "*cor. for alt.*" is additive in the case of the moon, and subtractive in the case of the sun.

Ex. 596. Prove that the perpendicular ascent or descent of a star is always quickest in the prime vertical. Illustrate the practical importance of this fact. *Honours*, 1877.

Ex. 597. Define the term "equation of equal altitudes." If t be the time shown by chronometer at the a.m. observation; a the estimated error of chronometer on the mean time at the place; e the equation of time (additive to M. T.); find an expression giving, approximately, the time by chronometer when the p.m. observation should be made. *Honours*, 1883.

Ex. 598. Explain the cause of twilight, and write down the

formulæ for determining its duration at a given place on a given day. *Honours*, 1870.

Ex. 599. Show how to compute the augmentation of the moon's semidiameter.

Ex. 600. September 29th, 1887, in longitude 168° W., the observed meridian altitude of the moon's lower limb was 48° 30′ 30″ (zenith south of the moon), index correction − 4′ 10″, the height of the eye above the sea 20 feet; find the true altitude of the moon's centre.

Ex. 601. At what time will Algenib pass the meridian 15° 28′ E. on August 20th, 1887?

Ex. 602. If twenty-four mean solar hours = 24h. 3m. 56·555s. sidereal time, express 3 days 15h. mean solar time in sidereal time.

Ex. 603. Orion's Belt being in the Equator, and having about 5h. 30m R. A., during what part of the night will it be visible at the vernal and autumnal equinoxes in latitude 50° 22′ N.?

Ex. 604. Given the hour angle of a heavenly body = 2h. 48m. 17s., declination 25° 28′ 43′ N., and altitude 48° 30′ (west of the meridian), calculate the azimuth and latitude.

Ex. 605. Define longitude of a place, and state the several methods by which it may be found on shore or at sea.

Ex. 606. Give definitions of latitude on the sphere and on the spheroid.

Ex. 607. Why is the mean solar year shorter than the sidereal year, and a mean solar day longer than a sidereal day?

Ex. 608. Having given the latitude of a place, and the time of rising of a heavenly body, show how to find the altitude when on the meridian.

Ex. 609. Eddystone lighthouse is 130 feet high, and is just visible from the mast of a vessel 53 feet above the water. Required the distance of the vessel from the Eddystone.

Ex. 610. Given the sun's declination 22° 7′ 52·8″ S., and his R. A. 16h. 38m. 25·49s., find his longitude and the obliquity of the ecliptic.

Ex. 611. In latitude 27° 35′ S. the sun bore N. 74° 42′ E. at 9 a.m. Required his altitude and declination.

Ex. 612. If the altitude of the sun when due W. be 25° 30′, and when on the six o'clock hour circle be 12° 29′ 50″, find the latitude and declination.

Ex. 613. The sun set W. 7° 29′ N. when his declination was 7° 4′ N. Required the latitude.

Ex. 614. If on August 17th, 1887, Spica, R. A. 13h. 19m. 15·05s., declination 10° 34′ 18·5″ S., set $2\frac{1}{2}$ hours before Arcturus, R. A. 14h. 10m. 30·94s., declination 19° 46′ 24·37″ N., find the north latitude.

Ex. 615. If the sun's altitude on the six o'clock hour circle be 7° 11′ 18·6″, and he set at 6h. 45m. 52·7s. apparent time, required the latitude and declination.

Ex. 616. When the sun's declination was 9° 21′ he set 1h. 17m. 13·3s. after he passed the prime vertical. Required the latitude.

Ex. 617. Given the sun's meridian altitude 48° 58′ 35″, bearing south, and altitude at six o'clock 7° 11′ 18·6″, find the latitude and declination.

Ex. 618. The altitude of a star when due E. was 15°, and when due S. was 45°; find the latitude.

Ex. 619. 1887, September 9th, at noon in longitude 4° 7′ 16·5″ W., a suspended sphere was observed to cast a shadow on a horizontal plane whose length was to its breadth as the diagonal to the side of a square. Required the place of observation.

Ex. 620. The extremity of the shadow of a steeple was 160 feet from its base at midwinter, and 60 feet at the equinox. Find the latitude if the maximum declination of the sun be 23° 28′.

Ex. 621. At the Plymouth Navigation School, latitude 50° 22′ 25″ N., find the difference in length of the longest and and shortest days, taking the sun's declination on the tropics as 23° 28′.

Ex. 622. In what north latitude will the day be two hours longer when the sun's declination is 19° 8′ 5″ N. than when he is 8° N.?

Ex. 623. In what latitude will the difference between the longest and shortest days be 8h. 25m. 54s.?

Ex. 624. In what latitude will the longest day be just $2\frac{1}{12}$ times longer than the shortest?

Ex. 625. The angular distance of Aldebaran from the moon's centre at 3h. 40m. at a certain place was 66° 14′; at Greenwich at noon and at 3h. the distances of the same object were 65° 9′ 30″ and 66° 41′ 30″ respectively. Determine the longitude of the place.

Ex. 626. What will be about the distance of a star whose parallax is $2''$?

Ex. 627. Two declination circles, P A, P B, make with each other a small angle ϵ at P, and from a point A (declination δ) in one of them an arc A B of a great circle is let fall perpendicularly on the other. Show that the difference of declination of A and B $= \sin 2\delta \,.\, \sin^2 \frac{\epsilon}{2}$ nearly.

Ex. 628. Given the latitudes and longitudes of two places where the inclination of the magnetic needle is nothing, to find the point of the terrestrial equator which is cut by the magnetic equator, supposing it a great circle of the earth.

Ex. 629. Two places in latitude 45°, and whose difference of longitude is 90°, are at two-thirds of the distance of places on the equator, with the same difference of longitude. Prove this.

Ex. 630. In a given latitude and longitude a vertical plane declines a ° from the south towards the west; find the place to whose horizon the plane is parallel.

Ex. 631. Two known stars are seen at a given place A on the same vertical, when at another place B they are rising together. Find the latitude and longitude of B.

Ex. 632. The mean time being four hours, find the corresponding sidereal time, having given the sun's mean daily motion 59′ 8·33″, and the R. A. at the preceding noon (mean) 144°.

Ex. 633. If the sidereal time at mean noon were 16h. 20m. 48s., what was the error of a watch at two o'clock, when a sidereal clock was at 18h. 21m., the sun's mean motion in longitude being 59′ 8·33″ in a mean solar day?

Ex. 634. The sun's apparent R. A. at mean noon, Greenwich time, on June 1st, 1860, was 4h. 38m. 18·96s., and on June 2nd, 4h. 42m. 24·73s.; find the sun's apparent R. A. at 11h. 20m. a.m. on June 2nd at a place 54° E.

And if the sun's R. A. at apparent noon on June 2nd be 4h. 42m. 24·34s., find approximately the equation of time.

Ex. 635. The right ascensions of two stars which crossed the meridian of Greenwich at mean noon yesterday and to-day respectively are 72° 23′ 21·15″ and 73° 51′ 29·55″; find the mean time at which a star whose R. A. is 317° 21′ 0·6″ crossed the same meridian yesterday.

Ex. 636. If Jupiter revolves round the sun in 4320 of our days, and round his own axis in ten hours, find by how much *his* mean solar day exceeds his sidereal day.

Ex. 637. Two rods, the one 6 feet, the other 8 feet, are placed on a given day, perpendicularly to the horizon, at a distance of 20 feet from each other. During the forenoon the extremity of the shadow of the first rod falls at the base of the second. In the afternoon the extremity of the shadow of the second falls at the base of the first. Required the latitude of the place, and the azimuth of one rod seen from the other.

Ex. 638. Given the point of the horizon at which the centre of the sun's disc rises, and the altitude of the point at which it crosses the meridian, find the time of year and the latitude of the place.

Ex, 639. Given the sun's altitude and azimuth, and the latitude of the place: to find the declination and hour of the day.

Ex. 640. Given the latitude of the place and the sun's declination, find the time when the hour angle from noon and the sun's azimuth from the south are equal.

Ex. 641. Given the sun's altitude and declination, and the sum of the azimuth and hour angle: to determine the latitude.

Ex. 642. Given the latitude of the place and the sun's declination; find at what time of the day the azimuth of the sun increases slowest.

Ex. 643. Given the sun's diameter and the latitude of the place, determine the declination when the time the sun takes to rise is a minimum.

Ex. 644. At a place the north latitude of which is 54°, and at a time of the year when the sun's north declination is 18°, show that if h is the hour angle which the sun makes with the meridian at the moment of sunrise, then cos. $h = \frac{1}{\sqrt{5}}$, and the sun rises shortly after four o'clock.

Ex. 645. Find the length of the shadow of a man 6 feet high in latitude 60° at 8 a.m. on the 21st of March; find also the direction in which the shadow points.

Ex. 646. In latitude 45°, at the equinox, find the time occupied by the sun in rising, assuming his diameter to be 30′.

Ex. 647. If h be the hour angle, and A the azimuth of a star at the instant when its altitude is equal to the latitude (ϕ) of the place of observation, show that

$$\cos. h = \tan. \phi \,.\, \tan. (45 - \tfrac{1}{2}\delta)$$
$$\sin. \tfrac{1}{2} A = \sec. \phi \,.\, \sin. (45 - \tfrac{1}{2}\delta),$$

δ being the declination.

Ex. 648. At a place in latitude ϕ, a wall of height h has an azimuth of a° to the east of south; show that at the time of the equinox the wall casts no shadow at $\frac{1}{15}\tan.^{-1}(\sin.\phi\,.\,\tan.a)$ hours before noon; and at noon the breadth of the shadow is $h\,.\,\tan.\phi\,.\,\sin.a$.

Ex. 649. If r be the horizontal refraction, show that the point of the compass where the sun rises is shifted by it

$$\frac{\sin.\phi}{\sqrt{\cos.(\phi-\delta)\,.\,\cos.(\phi+\delta)}}\,.\,r, \text{ where } \phi \text{ is the latitude.}$$

Ex. 650. At a place (lat. l) in the Arctic circle, the sun will remain above the horizon at the summer solstice for $\frac{365\frac{1}{4}}{\pi}\,.\,\cos.^{-1}.(\cos.l\,.\,\text{cosec.}\,\omega)$ days, neglecting the eccentricity of the earth's orbit.

Ex. 651. Find the moon's distance from the earth when her equatorial horizontal parallax is 56′ 30″.

Ex. 652. The latitude of a place A is 40° N.; of B, 50° N., and to the east of A; and their distance from each other is 20°; the longitude of A is 15° E. Required the latitude and longitude of another place C to the north of, and 20° distant from, A and B.

Ex. 653. Two known stars are in the same vertical circle, and the altitude of one of them is taken. Required the latitude of the observer. *A.* 1864.

Ex. 654. 1887, May 27th, a.m. at ship in latitude 27° 20′ 45″ N., the mean of the observed altitudes of the sun's upper limb was 33° 25′ 20″; index error + 2′ 25″; height of eye 23 feet. Time by chronometer 26d. 20h. 15m., which was slow on the same morning of Greenwich mean time 1h. 2m. 5s. The mean of the distances measured (at the same instant as the altitudes) between the sun's nearest limb and an object to the northward of the sun and in the horizon was 87° 2′ 40″; index error of the sextant used − 2′ 50″. Required the longitude and true bearing of the object.

Ex. 655. If the same object bore by compass due north, variation of the compass 20° 50′ W., what is the deviation, taking the results of the last question?

Ex. 656. In latitude 66° N. the sun's amplitude was observed to be three times his declination. Required the day of the month, supposing his declination to be that at noon, and to *remain* unchanged throughout the day.

Ex. 657. In the spring of the year 1887 the sun rose at 5h. 14m. a.m. Two hours after the shadow of a steeple was 76 feet, when the sun was due east. Required the height of the steeple, the sun's altitude, and the latitude of the place of observation.

Ex. 658. From the top of a mountain h miles high, the visible horizon appeared depressed a°; it is required to show that if d be the distance of the visible horizon, and D the diameter of the Earth,

$$D + h = h \,.\, \cot.^2 \frac{a^\circ}{2}$$

Ex. 659. If the depression of the visible horizon be $2^\circ\ 2'\ 11{\cdot}14''$ from the top of a mountain $2\frac{1}{2}$ miles above mean sea level, find the diameter of the Earth.

Ex. 660. On the 1st of June, 1887, the time from six a.m. to the time when the sun was due east was equal to the sun's amplitude at rising. If the declination for the day be assumed to be that at noon, find the latitude of observation.

Ex. 661. On May 26th, 1887, the altitude of the sun at six a.m. on the meridian of Greenwich was one-third of the latitude of the place of observation. Find the latitude.

Ex. 662. In latitude $25^\circ\ 17'$ N. find the time from its transit when Pollux, whose declination is $28^\circ\ 20'\ 15{\cdot}56''$ N., will appear stationary in azimuth, the period during which the star increased in azimuth after rising, and the number of degrees it moved backward.

Ex. 663. On the longest day in the year 1887, in south latitude, it was observed that the sun's bearing from west was equal to the apparent time at ship when his altitude was 40°. Required the latitude and the apparent time at ship, supposing the sun's declination to be the same as at noon.

Ex. 664. Dublin is in $53^\circ\ 21'$ N. latitude, and $6^\circ\ 19'$ W. longitude; Pernambuco is in $8^\circ\ 13'$ S. latitude, and $35^\circ\ 5'$ W. longitude. Required the day when he is on the horizon of both places at the same instant.

Ex. 665. Given the sun's declination on 19th June, 1887, $= 23^\circ\ 26'\ 1{\cdot}8''$; 20th June $= 23^\circ\ 26'\ 45{\cdot}7''$; 21st June $= 23^\circ\ 27'\ 4{\cdot}9''$; and 22nd June $= 23^\circ\ 26'\ 59{\cdot}2''$; find

(*a*) The declination on 20th day, 18h.

(*b*) The time when the sun has his greatest declination.

(*c*) The maximum declination.

Ex. 666. Find the hour angle and latitude of a place, having given three altitudes of an object taken near the meridian, and the interval of time elapsed between each two.

Ex. 667. Two stars, whose right ascensions are 8h. 52m. 5s. and 7h. 38m. 9s., are rising together at a place in north latitude, their amplitudes being E. 40° 30′ N. and E. 17° N. respectively. Find the latitude of the place and the declinatian of the two stars. *Honours*, 1884.

Ex. 668. If you were wrecked on a desert island, not knowing its position nor day of the year, with only writing materials, a watch, sextant, "Nautical Almanac" and Tables, how would you determine your position and the date?

Ex. 669. In what north latitude, and on what days in the year 1887 was the sun's declination double of his altitude at six, when the difference of the sines of his meridian altitude and midnight depression was equal to the sine of half the latitude, neglecting the change in declination?

ANSWERS.

Exercise I. (pages 12—14).

(20) Declination 23° 42′ 6″ N. | Right ascension 61° 19′ 19″.

Exercise II. (pages 29—31).

(24) 140°
Index error — 4′ 56·6″
Semidiameter 15 48·3
(25) Semidiameter 15 43·3
Index error — 38 43·3
(26) + 1 5
(28) Index error + 15″
Semidiameter 16′ 17·5″
(32) 14° 0′ 0″
(39) 33·8
(40) 1 42

Exercise III. (pages 41—44).

(52) April, 6d. 10h. 6m. a.m.
3 3 36s.
(53) One day.
(55) 3h. 57m. 51s.
40° 3′ 45″
(56) 30° E. or 150° W.
(57) 2d. 10h. 9m. 31·675s.
(58) 3 1 12
(61) 59′ 8·3298″
(64) 23h. 56m. 4·0906s.
(65) 59′ 8·3298″ E.
(66) 360° 59 8·33
(68) 13h. 20m. 36·159s.
(69) 20 9 15·35
(70) 18 12 8·922
(71) 18 6 11·56
(72) (*a*) 1 37 25·3
(*b*) 43 46
(*c*) 11 21 13
(*d*) 2 46 41·6
(*e*) 5 58 39·3
(*f*) 6h. 41m. 41·2s.
(73) (*a*) 92° 5′ 15″
(*b*) 183 48 30
(*c*) 6 52 30
(*d*) 138 4 30
(*e*) 4 30
(*f*) 238 21 45
(74) March, 28d. 6h. 11m. 21·73s.
(75) 199° 36′ 15″
6h. 6m. 20·26s. a.m.
(76) Feb., 28d. 16h. 53m. 55s.
(77) Greenwich time, 4h. 46m. 46s. afternoon.
Sydney time, 2h. 51m. 44s. next morning.
(78) March, 16d. 5h. 16m. 5·66s.
(79) Aug., 31d. 23h. 13m. 33·21s.
(80) *Rate*, 10·31s. gaining.
April, 14d. 20h. 50m. 59·26s.
(81) *Rate*, 10·2s. losing.
Oct., 15d. 9h. 29m. 13·17s.

EXERCISE IV. (pages 68—72).

(89) 4′ 53·7″, 4′ 59·5″
5 5·2
(92) 2° 10 16·8
(102) 6·1
(103) 10·6
(105) 10·7
(106) ϵ = ·0168″

(115) 241,144 miles
(118) 53′ 30·1″
(119) 234,883 miles
(120) 48′ 42·8″
(124) 48° 45 0·75 N.
(127) 386 : 1

EXERCISE V. (pages 90—92).

(142) Sept., 1d. 13h. 8m. 19s.
Dec. = 8° 6′ 51·9″ N.
Eq. time = + 14·2s. to M.T.
(143) March, 20d. 9h. 41m. 53s.
Dec. = 0° 0′ 28·7″ S.
Eq. time = + 7m. 30·48s. to A. T.
(144) R. A. = 5h. 30m. 29·05s.
Dec. = 18° 53′ 6·8″ N.
Semi. = 15 40·8
Par. in alt. = 35′ 7·2″
(146) 27° 23′ 39·2″
(147) 41 22 45·8

(148) 58° 41′ 32·2″
(149) 53 3 18·1
(150) 34 45 32·4
(151) 26 54 28·5
(156) Dec. = 17° 3′ 7·9″ N.
Eq. time = 3m. 39·48s. + to M. T.
Dec. = 15° 37′ 13·5″ S.
Eq. time = 16m. 18·32s. + to M. T.
(159) 8 min. — from M. T.
(163) Moon's hor. semidiam. = 16′ 15·4″

EXERCISE VI. (pages 99—102).

(173) 6h. 10m. 40s.
(176) 0 3 29 a.m.
11 59 33 p.m.
(177) 10 59 49 p.m.
32° 58′ 49″ S.
(178) 0h. 43m. 43s. a.m.
12° 30′ 59″ N.
(179) 2h. 38m. 9s. a.m.
Capella and Rigel
(180) 10h. 47m. 34s. p.m.
15° 43′ 44″ S.
(181) 19h. 29m. 37s.
(182) From α Cephei to Algenib
(183) From ω Piscium to ο Piscium
(184) From α Coronæ to α′ Hercules

(185) 7h. 6m. 20s. a.m.
(186) 10 48 9
(188) 6 0 0
(189) 6 0 0 p.m.
(190) 10 16 45
(191) 13 51 13
(192) 0 25 36
(193) 18 36 41·6
(194) 17 23 14·8
(195) 14 2 40·8
(196) 1 27 51·3 a.m.
1 25 53·4 p.m.
(197) 3 49 44
45° 26′ 16″ N.
(198) 78 2 10 N.
10h. 12m. 25s. a.m.

Exercise VII. (pages 115—118).

(206)	2°	31′	57″	S.
(207)	39	50	50	S.
(208)	74	55	24	N.
(209)	2	16	22	N.
(210)	27	29	54	S.
(211)	54	53	18	N.
(212)	37	4	53	N.
(213)	36	53	39	S.
(214)	42	10	38	S.
(215)	83	21	8	S.
(216)	23	8	50	N.
(217)	42°	20′	56″	S.
(218)	36	44	5	S.
(219)	28	44	35	N.
(221)	55			
(223)	45 each			
(226)	Lat. = dec.			
(227)	68°	0′	10″	N.
(228)	76	8	49	N.
	or 43	22	53	S.
(232)	48	32	N.	

Exercise VIII. (pages 129—131).

(236)	Reduction	52°	15′	49″	N.
	Direct	52	15	37	N.
(237)	Reduction	41	35	37	N.
	Direct	41	35	38	N.
(238)	Reduction	51	55	9	S.
	Direct	51	55	45	S.
(239)	Reduction	43	48	27	S.
	Direct	43	48	27	S.
(240)	Reduction	39°	20′	1″	N.
	Direct	39	19	56	N.
(241)	Reduction	45	5	6	S.
	Direct	45	5	12	S.
(242)	Reduction	29	0	29	S.
	Direct	29	0	28	S.
(243)	Reduction	59	24	53	S.
	Direct	59	24	52	S.

Exercise IX. (pages 138—140).

(252)	49°	56′	45″	N.
(253)	52	25	4	N.
(254)	66	23	29	N.
(255)	36	55	39	N.
(256)	41	57	34	N.
(257)	77°	28′	0″	N.
(259)	55	9	23	N.
(263)	31	50	13	N.
(265)			43″	
(267)	30	17	33	N.

Exercise X. (pages 159—164).

(274)	167°	58′	0″	W.
(275)	56	0	22	E.
(276)	0	7	38	W.
	0	15	55	W.
(277)	179	55	11	W.
		12	34	too little.
(278)	101	42	4	E.
(279)	93	54	36	E.
(280)	27	57	14	E.
(281)	143	41	48	W.
	143	33	29	W.
(282)	0°	0′	3″	W.
(283)	58	52	30	W.
(284)	173	40	44	W.
		4	9	too great.
(285)	168	33	49	W.
(286)	41	35	57	W.
		14	55	too great.
(287)	37	25	12	W.
	37	53	44	W.
(299)	53	46	27·7	
(302)		8	59	

(303) 12h. 21m. 55s.
(304) 1 48 21 a.m. 12th
(305) 8′ 3″
(306) 8° 8′ 53″ W.
(307) As 1 10·2

EXERCISE XI. (pages 176—181).

(312) 4·4s. gaining
(313) 4·78s. losing
(314) 11m. 51s. fast
(315) 54 31 slow
(316) 14 43 fast
(317) 5 51 slow
(318) 1 5 fast
(319) 9 54 fast
3·14 gaining
(320) 4h. 48m. 21·4s. fast A. T. S.
4 40·3 slow G. M. T.
(321) 2 20 8·4 fast A. T. S.
14 15·7 slow G. M. T.
(322) 4 19 32·7 fast A. T. S.
13 30 fast G. M. T.
(323) 2·5 slow A. T. S.
1 0 39·5 fast G. M. T.
(324) 3h. 50m. 48·6s. slow A. T. S.
2 3·9 fast G. M. T.
(325) 3 32·7 slow A. T. S.
5 19 53·5 fast G. M. T.
(326) Slow mean time place
1h. 6m. 43·7s.
Slow M. T. G. 4h. 37m. 23·7s.
April 10th slow M. T. S.
1h. 6m. 6·7s.
(327) 2h. 48m. 40s.
(328) Jan. 31, 2h. 11m. 30s.
(330) 9m. 6·5s.
(331) 48 45·5 fast G. M. T.
3 41·22 gaining
(337) 6·56
(339) 1h. 38m. 43s.
(340) 3m. 17·2s. fast G. M. T.

EXERCISE XII. (pages 186—188).

(343) Compass err. 30° 42′ 9″ W.
Deviation 12 22 9 W.
(344) Compass err. 19 47 E.
Deviation 19 50 13 W.
(345) Deviation 9 1 17 E.
(346) Compass err. 10 30 36 E.
Deviation 35 49 21 E.
(347) Compass err. 26 1 56 W.
Deviation 1 13 4 E.
(348) Compass err. 40 23 W.
Deviation 3 30 23 W.
(349) Compass err. 14° 8′ 5″ W.
Deviation 12 11 55 E.
(350) Compass err. 7 48 21 E.
Deviation 2 1 39 W.
(351) Variation 22 23 54 W.
(352) Cor. for course 22 40 50 E.
Deviation 13 55 50 E.
(353) North
(356) Comp. bear. S. 73 6 21 W.
(357) $\frac{1}{2}\sqrt{2+\sqrt{2}}\,.\cos. l$

EXERCISE XIII. (pages 193, 194).

(362) 3h. 59m. 47·1s. a.m.
(363) 8 34 33·6
(365) 53° 27′ 22·3″ N.
(366) 41 23 47·9
(367) 41 23 47·9
(371) 48° 54′ 12·5″ N.
(375) 61 46 54·6 N.
(376) 3m. 7·8s.
(379) (*a*) At the equinoxes
(*b*) On the equator